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THE SECRET OF FLIGHT

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Chapter 1

Short Story of Flight

> We call it theory when we know much about something but nothing works, and practice when everything works but nobody knows why. (Einstein)
>
> Of all the forces of Nature, I would think the wind contains the greatest amount of power. (Abraham Lincoln)

1.1 Why is it Possible to Fly?

What keeps a bird or airplane in the air? How can the flow of air around a wing generate large *lift* L (balancing gravitation) at small *drag* D (requiring forward thrust) with a *lift to drag ratio* also referred to as *finesse* $\frac{L}{D}$ ranging from 10 for short thick wings to 70 for the long thin wings of extreme gliders, which allows flying at affordable power for both birds and airplanes?

An albatross with finesse $\frac{L}{D} = 50$ can glide 50 meters upon losing 1 meter in altitude. A $400\,ton$ Airbus 380 takes off at a wing lift of $1\,ton/m^2$ from maximal engine thrust of 40 tons with $\frac{L}{D} = 10$. The dream of human-powered flight came true in 1977 on $60\ m^2$ wings generating a lift of $100\,kp$ at a thrust of $5\,kp$ (thus with $\frac{L}{D} = 20$) at a speed of $5\,m/s$ supplied by a $0.3\,hp$ human powered pedal propeller as shown in Fig. 1.1.

How is this possible? Is it a miracle? Is there a theory of flight and what does it say? What is the dependence of lift and drag on wing form, wing area, angle of attack (the tilting of the wing in the flow direction) and speed? Can engineers compute the distribution of forces on an Airbus 380 during take-off and landing using mathematics and computers, or is model testing in wind tunnels the only way to figure out if a new design will work? Let's see what media says.

New York Times informs us in 2003 [130] under the slightly scaring headline *STAYING ALOFT: What Does Keep Them Up There?*:

- *To those who fear flying, it is probably disconcerting that physicists and aeronautical engineers still passionately debate the fundamental issue underlying this endeavor: what keeps planes in the air?*

The authority NASA dismisses on its website [117] all popular science theories for lift, including your favorite one, as being incorrect, but refrains from presenting any theory claimed to be correct and ends without helping the curious with: *To truly understand the details of the generation of lift, one has to have a good working knowledge of the Euler Equations*.

Is it possible that NASA cannot explain what keeps planes in the air? Yes, it is possible: birds fly without explaining anything. Is it not only possible but even true that NASA does not know? We leave this question to the reader to answer after digesting the book.

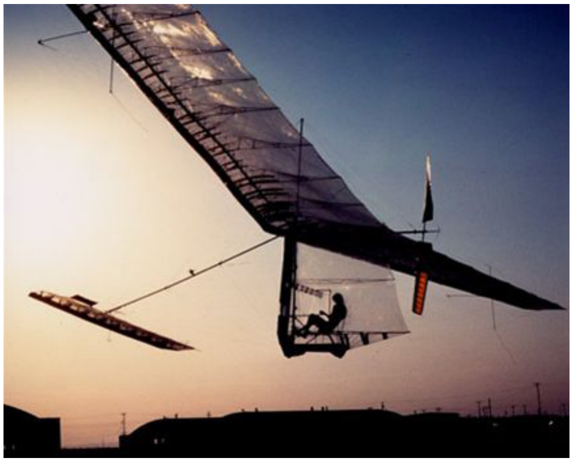


Figure 1.1: The dream of human-powered flight came true in 1977: The Gossamer Condor constructed by Paul MacCready (wing area of 60 m^2 and weight of 32 kg) flew the first figure-eight, a distance of 2.172 meters winning the first Kremer prize of 50.000 pounds offered by the industrialist Henry Kremer in 1959.

Text books tell a different story than New York Times and NASA, by claiming: Yes, there is a solid mathematical theory of flight explaining what keeps planes in the air, which was developed 100 years ago by

- the physicist Prandtl, named the father of modern fluid mechanics,

- the mathematicians Kutta and Zhukovsky, named fathers of aviation.

These fathers suggested in 1904 that both lift and drag of a wing result from an effect of *viscosity* retarding fluid particles in a thin boundary layer from free stream to zero *no-slip* relative velocity on the wing surface, capable of changing the global features of the flow from the

- zero lift and drag of idealized unphysical *inviscid potential flow*,

to the

- large lift and small drag of real physical *turbulent slightly viscous flow*.

The unphysical feature of potential inviscid flow of having zero drag (and lift making flight in particular impossible) named *d'Alembert's paradox* discovered in 1752, was thus claimed by Prandtl in 1904 to be a result of allowing fluid particles to glide or *slip* on the wing surface without friction. Prandtl thus suggested that drag was caused by a no-slip boundary layer. Kutta and Zhukovsky complemented by obtaining lift by modifying potential flow with a flow circulating around the wing section and thus obtained lift from *circulation*, assuming that the circulation somehow was created by the boundary layer.

This became the basic postulate of modern fluid mechanics: In mathematical modeling of fluid flow it is necessary to use a *no-slip boundary condition* accompanied by a *viscous boundary layer*: Both lift and drag originate from the boundary layer; without a boundary layer there will be no lift and no drag. The current text book theory of flight is basically a 100 year old Kutta-Zhukovsky-Prandtl theory with Kutta-Zhukovsky supplying lift (without drag) and Prandtl drag (without lift).

Prandtl used a no-slip boundary condition in 1904 to handle d'Alembert's paradox which had paralyzed theoretical fluid mechanics from start, but in doing so Prandtl sent fluid mechanics from an 18th century paralysis of zero lift and drag into a 20th and even 21st century computational mathematics impossibility of resolving very thin boundary layers.

1.2 New Theory of Flight

In this book we present a mathematical theory of flight based on computing and analyzing turbulent solutions of the incompressible Navier-Stokes equations for

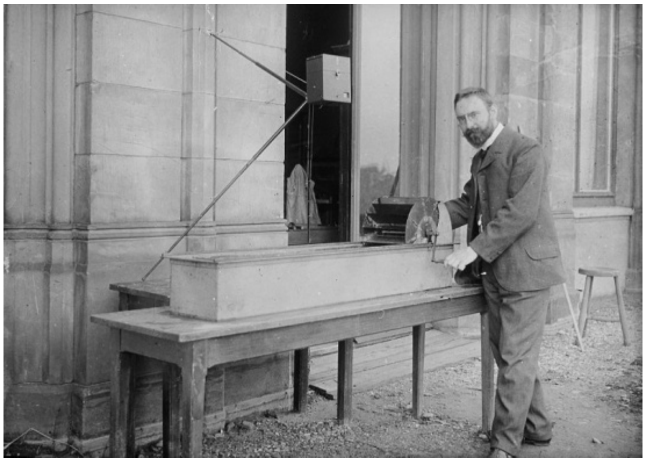


Figure 1.2: Prandtl formulating the credo of modern fluid dynamics in 1904: *I have now set myself the task to investigate systematically the laws of motion of a fluid whose viscosity is assumed to be very small...A very satisfactory explanation of the physical process in the boundary layer [Grenz-schicht] between a fluid and a solid body could be obtained by the hypothesis of an adhesion of the fluid to the walls, that is, by the hypothesis of a zero relative velocity between fluid and wall.*.

slightly viscous flow subject to slip boundary conditions. We thus break the no-slip spell of Prandtl and we do this by a new resolution of d'Alembert's paradox identifying the unphysical aspect of potential flow to be instability rather than violation of no-slip, understanding that slip correctly models the very small *skin friction* of slightly viscous flow.

We focus on subsonic flight as the most basic, mysterious and elegant form of flight practiced by birds, propeller airplanes and jet airplanes in take-off and landing, in which case the flow of air is nearly *incompressible*. We refer to this form of flight as *gliding flight*.

To fly at supersonic speed like a jet driven rocket in compressible flow is not gliding flight and no aerodynamics wonder. But like an albatross stay gliding for days without flapping the wings, is a miracle, and this is the miracle we will explain. Not by reference to exotic difficult mathematics which nobody can understand, but by exhibiting certain fundamental aspects of slightly viscous incompressible flow which can be understood without any tricky physics or mathematics.

We will thus identify the primary mechanism for the generation of substantial lift at the expense of small drag of a wing, and discover it to be fundamentally different from the classical circulation theory of lift by Kutta-Zhukovsky and the boundary layer separation theory of drag by Prandtl.

The essence of the new flight theory can be grasped from Fig. 1.3 depicting the following crucial features of the pressure distribution around a section of a wing with (H) indicating high pressure and (L) low pressure:

- Left: potential flow with leading edge positive lift from (H) on lower wing surface and (L) on upper, cancelled by trailing edge negative lift from (L) on lower and (H) on upper surface, giving zero lift. Potential flow does not cause *downwash* redirecting the flow downwards and thus no lift by reaction.

- Middle: instability at trailing edge separation with low pressure swirling flow reducing (H) on upper surface,

- Right: true flow with (H) and (L) switched at trailing edge by separation instability giving substantial upward lift as a reaction of downwash redirecting incoming flow downwards.

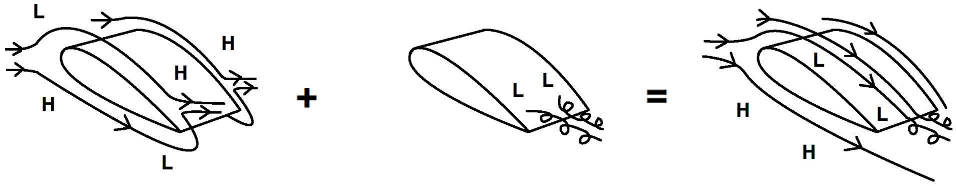


Figure 1.3: Correct explanation of lift by perturbation of potential flow (left) at separation from physical low-pressure turbulent counter-rotating rolls (middle) changing the pressure and velocity at the trailing edge into a flow with downwash and lift (right).

The rest of the book can be seen as an expansion and explanation of this picture with the following key questions to be answered:

- Why does the air stick to the upper wing surface without separation and thus cause lift by low-pressure suction?

- Why does the air not turn around the trailing edge like potential flow destroying lift?

We shall see that the miracle of gliding flight results from a happy combination of all of the following features of subsonic flow of air around a wing:

1. nearly incompressible with velocity well below the speed of sound,
2. sufficiently small viscosity, sufficient speed and size of the wing,
3. three dimensional separation at the trailing edge.

The requirement 2 can be expressed as a requirement of a sufficiently large ***Reynolds number*** $Re = \frac{UL}{\nu}$, where U is a characteristic fluid velocity ($10 - 100\,m/s$), L a characteristic length scale ($1 - 10\,m$ and ν the (kinematic) viscosity (about 10^{-5} for air).

Experiments recorded in [121] show that the finesse $\frac{L}{D}$ of a typical wing sharply increases from about 2 to 20 as a result of decreasing drag and increasing lift, as Re increases beyond a ***critical Reynolds number*** $Re_{crit} \approx 0.5 \times 10^6$. The drag reduction is in the literature described as *drag crisis* reflecting that it is poorly explained by Prandtl's theory. The increase of the finesse with a factor 10 is what makes gliding flight possible, and can be seen as a form of miracle as any factor 10 above normality. This book presents the mathematics of this miracle thereby turning it into science.

As a consequence, gliding flight in syrup is not possible (because of too large viscosity) and neithe in two space dimensions (because the three-dimensional separation is missing). We shall see that the classical flight theory of Kutta-Zhukovsky-Prandtl violates 2 and 3: Prandtl talks about a viscous flow, and Kutta-Zhukovsky about two-dimensional flow.

The result is that the flight theory of Kutta-Zhukovsky-Prandtl has nothing to do with reality. Aviation developed despite theory and not with the help of theory, just like real life is made possible by making a clear distinction from theoretical religious scholastics. To say one thing and do another may be a necessity of life, but is not the scientific method.

1.3 Butterfly in Brazil and Tornado in Texas

How can you prove that a butterfly in the Amazonas cannot set off a Tornado in Texas? Well, take away the butterfly and notice that tornados anyway develop.

We shall use the same strategy to show that lift and drag of a wing do not emanate from any boundary layer: In computational simulation we eliminate the

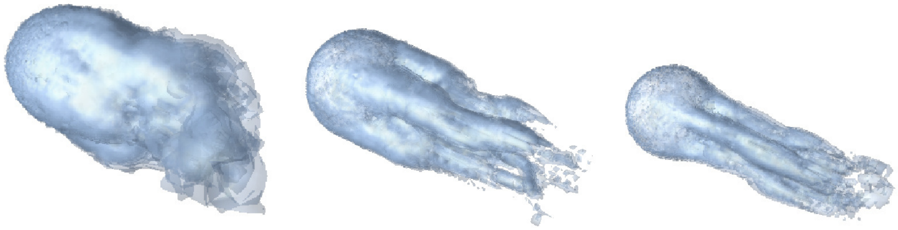


Figure 1.4: Illustration of the drag crisis in the flow around a sphere as Re passes $Re_{crit} \approx 0.5 \times 10^6$, with delayed separation and decreasing turbulent wake. Notice the separation at the crest for $Re < Re_{crit}$ with large drag and and the structure of the wake flow for $Re > Re_{crit}$ with small drag.

boundary layer by using a slip boundary condition, and we find that the computed lift and drag fit very well with experimental observation.

1.4 Lift and Drag without Boundary Layer

We thus show that Prandtl's explanation is insufficient by computationally solving the Navier-Stokes equations with slip boundary condition, which eliminates the boundary layer, and observing turbulent solutions with lift and drag in accordance with observation. We thus observe tornados without any butterfly: Lift and drag do not originate from a thin boundary layer. The central thesis of Prandtl underlying modern fluid mechanics cannot be correct. As a consequence much of the material propagated in text books of fluid mechanics needs to be fundamentally revised.

1.5 Why Slip is a Physical Boundary Condition

The slip boundary condition models the small boundary friction force of slightly viscous flow, understanding that Navier-Stokes equations can be solved with (small) friction force boundary conditions or no-slip velocity boundary conditions, and that the former choice is computationally more advantageous since there is no no-slip boundary layer to resolve. We will show that this allows accurate computation of both lift and drag for complex geometries using millions of mesh point, which is possible today, instead of the quadrillions required for boundary layer resolution, which may never be possible.

THE BUTTERFLY EFFECT
A BUTTERFLY'S WINGS HAS THE POTENTIAL TO CREATE TINY CHANGES IN
THE ATMOSPHERE THAT CAN ULTIMATELY CAUSE A TORNADO TO FORM
THE FLAPPING WING REPRESENTS A SMALL CHANGE IN THE INITIAL CON-
DITION OF THE SYSTEM, WHICH CAUSES A CHAIN OF EVENTS LEADING TO
LARGE-SCALE PHENOMENA. HAD THE BUTTERFLY NOT FLAPPED ITS
WINGS, THE TRAJECTORY OF THE SYSTEM MIGHT HAVE BEEN VASTLY
DIFFERENT.
SMALL VARIATIONS OF THE INITIAL CONDITION OF A NONLINEAR DYNAMICAL SYSTEM MAY
PRODUCE LARGE VARIATIONS IN THE LONG TERM BEHAVIOR OF THE SYSTEM
TORNADO
y
SENSITIVE DEPENDENCY ON INITIAL CONDITIONS
attractor C

Figure 1.5: Tornados in Texas are not caused by butterflies in Brazil.

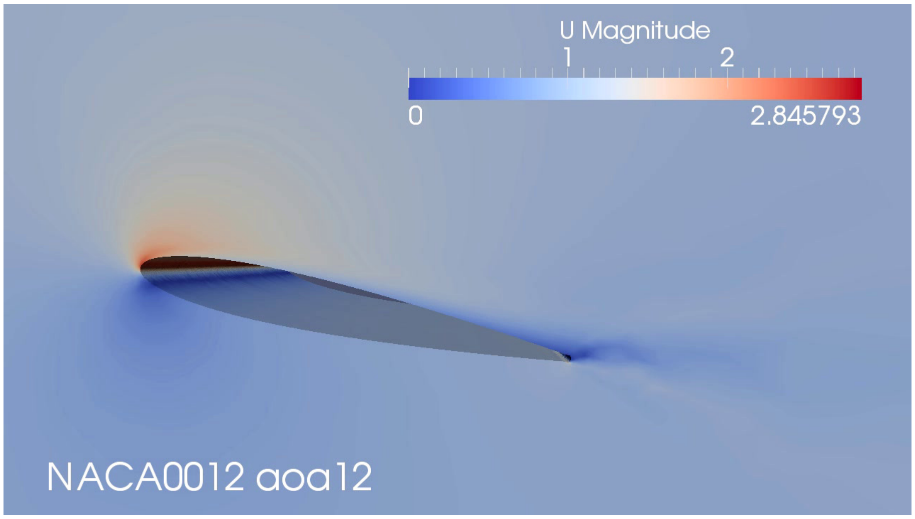


Figure 1.6: Lift and drag of a wing are not caused by a thin boundary layer.

1.6 Compute-Analyze-Understand

This book is based on computing solutions to the incompressible Navier-Stokes equations with slip using a computational method with automatic a posteriori error control in mean-value outputs such as lift and drag. The error control is based on duality measuring output sensitivity with respect to resolution of the Navier-Stokes equations in the fluid domain and the boundary conditions.

Computed solutions thus can be guaranteed to give correct output information (up to a tolerance depending on the computational work) and thereby offer a digital fluids laboratory or wind tunnel. This book shows that the secret of flight is revealed by combining the experimental facility of the digital laboratory with a bit of mathematical analysis.

The goal of the book is to make the reader say after browsing the book:

- I understand why it is possible to fly.
- I understand that whatever I would like to know about the fluid mechanics of flying can be answered by computation.

1.7 How to Say Goodbye in an Elegant Way

A part of the secret of flight is the three-dimensional slip separation at the trailing edge of real turbulent flow, which makes it possible for the flow to leave the wing without destroying the lift created by suction on the crest of the wing as is done by non-physical two-dimensional potential flow separation. The three-dimensional separation pattern is similar to the elegant swirling motion of the hand used by noble men at the royal court when backwards leaving from an audience with the king.

1.8 Flight and Turbulence

The reason it has taken so long time to develop a theory of flight describing the real physics, is that the flow of air around a wing is partly turbulent, and turbulence has been a main unresolved mystery of contiuum mechanics (as opposed to quantum mechanics): The complex and seemingly chaotic nature of turbulent flow defies description by analytical mathematics.

But today computers are powerful enough to allow computational simulation of turbulent flow which opens to demystify turbulence and thereby demystify flight. In [103] we demonstrated the new possibilties of both simulating and understanding turbulence and thus opened to a new theory of flight. The central experience is that turbulent flow can be simulated without resolving the physicallly smallest scales, which allows accurate prediction of mean-value outputs such as lift and drag with today affordable computer power.

Turbulent flow can be seen as Natures way of handling an intrinsic instability of all slightly viscous flow with the fluid constantly seeking a more stable configuration without ever succeeding and thus fluctuating on a range of scales from large scale more or less organized motion to smaller scale fully chaotic motion.

Zero lift and drag potential flow around a wing thus develops into a flow which at the trailing edge of the wing separates into turbulent flow which has big lift and small drag. Without turbulence it would be impossible to fly.

Turbulence consumes energy by internal friction or viscous dissipation into heat and thus flying requires input of energy, from flapping wings, propellers or jet engine or from upwinds for powerless gliders.

Turbulence is three-dimensional phenomenon, which makes aslo flying to three-dimensional phenomenon. The classical flight theories of Kutta-Zhukovsky-Prandtl are two-dimensional and do not describe real flight in three dimensions.

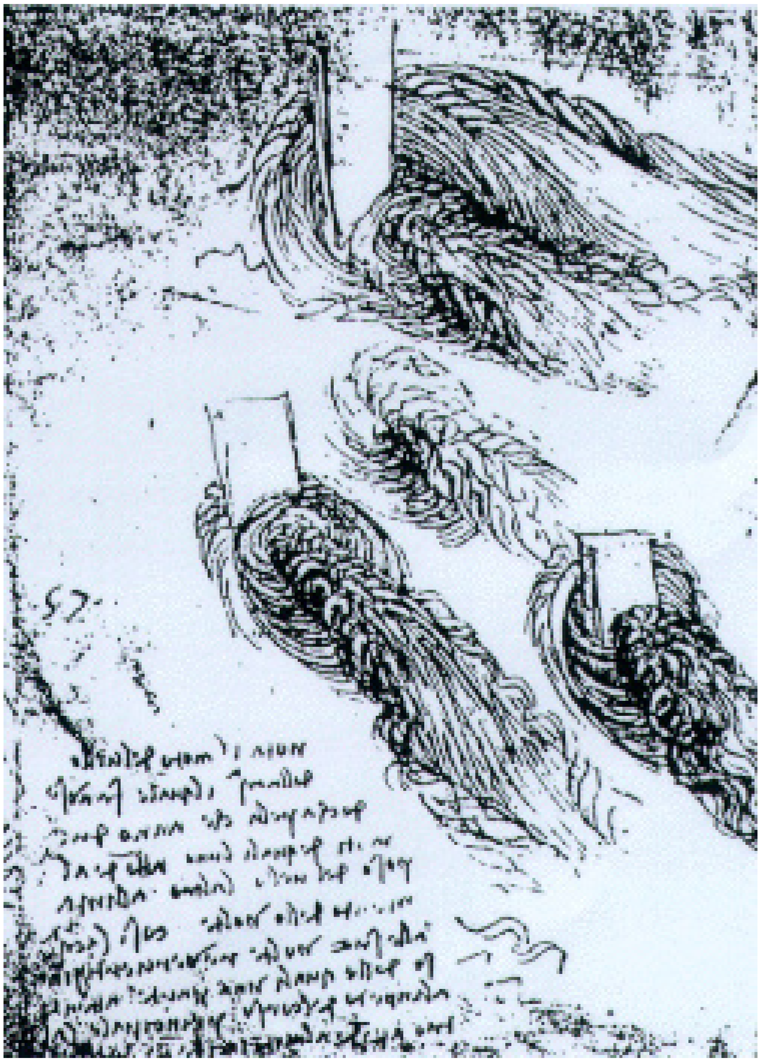


Figure 1.7: Turbulent flow depicted by Da Vinci. This picture properly viewed reveals the secret of flight.

Chapter 2

Short History of Aviation

> I feel perfectly confident, however, that this noble art will soon be brought home to man's general convenience, and that we shall be able to transport ourselves and families, and their goods and chattels, more securely by air than by water, and with a velocity of from 20 to 100 miles per hour. (George Cayley 1809)

2.1 Leonardo da Vinci, Newton and d'Alembert

Is it conceivable that with proper mathematics, humans would have been flying, at least gliders (without engine), several hundred years before this actually came true in the late 19th century? Well, let's face some facts.

The idea of flying, like the birds, goes back at least to Greek mythology about the inventor and master craftsman Deadalus, who built wings for himself and his son Icaros in order to escape from imprisonment in the Labyrinth of Knossos on the island of Crete.

Leonardo da Vinci made impressive and comprehensive investigations into aerodynamics collected into his *Treatise on the Flight of Birds* from 1505, and designed a large variety of devices for muscle-powered human flight. After extensive testing da Vinci concluded that even if both arms and legs got involved through elaborate mechanics, human power was insufficient to get off the ground.

Newton confirmed these experiences by calculating the lift of a tilted flat plate, representing a wing, in a horisontal stream of "air particles" hitting the plate from below, to obtain a disappointingly small (erronous) value of the lift.

Newton's result was further supported by d'Alembert's Mystery predicting that both the drag and the lift of a body traveling through air would be close to zero, clearly at variance with many early observations of birds flying long distances even without flapping their wings. d'Alembert built his computations of drag (and lift) on particular solutions to the Euler equations referred to as *potential solutions*, with the velocity given as the gradient of a potential satisfying Laplace's equation. But nobody could come up with any kind of resolution of the paradox before Ludwig Prandtl (1875-1953), called the Father of Modern Fluid Dynamics, in a short note from 1904 suggested a resolution based on boundary layer effects from vanishingly small viscosity, which still today remains the accepted resolution of the paradox. As already indicated, we shall below present computational evidence that Prandtl's resolution is not credible and instead put forward a new scientifically more satisfactory resolution.

2.2 Cayley and Lilienthal

Despite the pessimistic predictions by Newton and d'Alembert, the 29 years old engineer George Cayley (uncle of the mathematician Arthur Cayley) in 1799 sketched the by now familiar configuration of an airplane with fixed cambered wings and aft horisontal and vertical tails, and also investigated the characteristics of airfoils using a whirling arm apparatus. Cayley outlined his ideas about the principles of flying in *On Aerial Navigation* (1809). But Cayley did not produce any mathematical description of the motion of an aircraft and thus had no quantitative basis for designing airplanes. In 1849 Cayley built a large glider, along the lines of his 1799 design, and tested the device with a 10-year old boy aboard. The glider carried the boy aloft on at least one short flight.

The next major step was taken by the German engineer Otto Lilienthal, who made careful experiments on the lift and drag of wings of different shapes and designed various gliders, and himself made 2000 more or less successful flights starting from a little hill, see Fig 2.1, before he broke his neck in 1896 after the glider had stalled 15 meter above ground.

2.3 Kutta, Zhukovsky and the Wright Brothers

Stimulated by Lilienthal's successful flights and his widely spread book *Bird Flight as the Basis of Aviation* from 1899, the mathematician Martin Kutta (1867-1944) in his thesis from 1902 modified the erronous classical potential flow so-

Figure 2.1: Otto Lilienthal (1848-1896), some of the 137 known photos from 1891 to 1896. To document the development of his flight technique he was regulary joined by photographers during his flight practise (photos from Archive Otto-Lilienthal-Museum / `www.lilienthal-museum.de`).

lution by including a new term corresponding to a rotating flow around the wing with the strength of the vortex determined so that the combined flow velocity became zero at the trailing edge of the wing. This *Kutta condition* reflected the observation of Lilienthal that the flow should come off the wing smoothly, at least for small angles of attack. The strength of the vortex was equal to the *circulation* around the wing of the velocity, which was also equal to the lift. Kutta could this way predict the lift of various airfoils with a precision of practical interest. But the calculation assumed the flow to be fully two-dimensional and the wings to be very long and became inaccurate for shorter wings and large angles of attack.

The first successful powered piloted controled flight was performed by the brothers Orwille and Wilbur Wright on December 17 1903 on the windy fields of Kitty Hawk, North Carolina, with Orwille winning the bet to be the pilot of the *Flyer* and Wilbur watching on ground, see Fig 35.1. In the words of the Wright brothers from Century Magazine, September 1908: “The flight lasted only twelve seconds, a flight very modest compared with that of birds, but it was, nevertheless, the first in the history of the world in which a machine carrying a man had raised itself by its own power into the air in free flight, had sailed forward on a level course without reduction of speed, and had finally landed without being wrecked.

The second and third flights were a little longer, and the fourth lasted fifty-nine seconds, covering a distance of 852 feet over the ground against a twenty-mile wind." The modern era of aviation had started.

Figure 2.2: Orwille Wright (1871-1948) and Wilbur Wright (1867-1912) and the lift-off at Kitty Hawk, North Carolina, the 17th December 1903.

The mathematician Nikolai Zhukovsky (1847-1921), called the Father of Russian Aviation, in 1906 independently derived the same mathematics for computing lift as Kutta, after having observed several of Lilienthal's flights, which he presented before the Society of Friends of the Natural Sciences in Moscow as: "The most important invention of recent years in the area of aviation is the flying machine of the German engineer Otto Lilienthal". Zhukovsky also purchased one of the eight gliders which Lilienthal sold to members of the public.

Kutta and Zhukovsky thus could modify the mathemathical potential theory of lift of a wing to give reasonable results, but of course could not give anything but a very heuristic justification of their Kutta-Zhukovsky condition of zero velocity at the trailing edge of the wing, and could not treat realistic wings in three dimensions. Further, their modified potential solutions were not turbulent at all, so their calculations would seem merely like happy coincidences (knowing ahead the correct answer to obtain). We will return below in more detail to the basic problem of lift and drag of wings in turbulent flow.

Today computational methods open new possibilities of solving the equations for fluid flow using the computational power of modern computers. Thus, for the first time the mathematical fluid models of Euler and Navier-Stokes may come to a real use, which opens new revolutionary possibilities of computational simulation and prediction of fluid flow in science and technology. The range of possible applications is incredibly rich! For example, it is now becoming possible to simulate the turbulent flow around an entire aircraft and thus systematically investigate questions of stability and control, which caused severe head-ache for the Wright

Figure 2.3: Martin Kutta (1867-1944) and Nikolai Egorovich Zhukovsky (1847-1921).

brothers, as well as the designers of the modern Swedish jet fighter JAS Gripen. Actually, both the 1903 Wright *Flyer* airplane, with a forward canard instead of an aft tail, and the JAS are unstable and require careful control to fly. The instability of the fighter is intentional allowing quick turns, but the Wrights later replaced the canard with the conventional aft tail to improve stability. The stability of an airplane is similar to that of a boat, with the important design feature being the relative position of the center of gravity and the center of the forces from the fluid (center of buoyancy for a boat), with the center of gravity ahead (below) giving stability.

Figure 2.4: The 1903 Wright *Flyer* and JAS Gripen (JAS photo from `http://www.gripen.com/`.

It is remarkable that 400 years passed between Leonardo da Vinci's investigations and the largely similar ones by Lilienthal. Why did it take so long time from almost success to success? Can we blame the erronous mathematics of Newton and d'Alembert for the delay? Or was the reason that the (secret) writings of da Vinci were made public with a delay of 300 years? We leave the question open.

Chapter 3

Euler and Navier-Stokes

> However sublime are the researches on fluids which we owe to Messrs Bernoulli, Clairaut and d'Alembert, they flow so naturally from my two general formulae that one cannot sufficiently admire this accord of their profound meditations with the simplicity of the principles from which I have drawn my two equations ...(Euler 1752)

3.1 Model of Fluid Mechanics

The basic mathematical model of fluid mechanics (or fluid dynamics) consists of *Navier-Stokes equations* expressing *conservation of mass, momentum and energy* of a *viscous fluid*. Theoretical fluid mechanics can be viewed as the science of computing and analyzing solutions of the Navier-Stokes equations.

Water has a very small compressibility and air at speeds well below the speed of sound is nearly incompressible. The fluid mechanics of water and air at subsonic speeds is thus captured by the *incompressible Navier-Stokes equations* with the *Reynolds number* $Re = \frac{UL}{\nu}$ as an important flow characteristic, where U is a characterisitic flow speed, L a characteristic length scale and ν the kinematic viscosity. For water $\nu \approx 10^{-6}$ and for air $\nu \approx 10^{-5}$. The typical Reynolds number of an airplane may be 10^7, of larger birds 10^5 and of insects 10^3, while formally the *incompressible Euler equations* would describe flow at vanishing viscosity or infinite Reynolds number.

In the incompressible Navier-Stokes equations the equation expressing conservation of total energy as the sum of kinetic energy and heat energy, is decoupled from the equations expressing conservation of mass and momentum, and mass conservation can be reduced to requiring fluid velocities to be incompressible.

The incompressible Navier-Stokes/Euelr equations thus can be viewed to consist of two equations in the fluid velocity and pressure expressing

- Newton's 2nd Law (conservation of momentum),
- incompressibility.

Flight at speeds up to $300\,km/h$ of airplanes and larger birds is described by solutions of the incompressible Navier-Stokes equations at a Reynolds number of size 10^6 or larger, which with normalization to $U = L = 1$ translates to small viscosity of size 10^{-6} or smaller.

3.2 Boundary Conditions

The Navier-Stokes equations in the fluid domain are complemented by *boundary conditions* on the boundary of the fluid domain prescribing velocities or forces or combinations thereof. The following conditions imposed on the boundary of a solid object moving through a fluid have particular significance:

- no-slip : both normal and tangential velocities are put to zero,
- slip: normal velocity and tangential forces are put to zero.

We have already indicated the importance of the slip condition and will return to this crucial aspect below.

3.3 The Euler Equations

The incompressible Navier-Stokes equations were formulated in the 1830s as a generalization of the *incompressible Euler equations* for a fluid with vanishingly small viscosity (inviscid fluid) formulated by Leonard Euler around 1745 building on work by the son and father Bernouilli as a major new application of the tool of Calculus of Leibniz and Newton.

Euler quickly identified a class of solutions to his equations named *potential solutions* with the fluid velocity given as the gradient of a harmonic function or potential, allowing analytical solution of a wide class of basic flow problems, with a slip boundary condition. The hope of quick progress was manifested as the 1749

Prize Problem of the Berlin Academy of Science asking for computation of the drag of a body moving through an inviscid and incompressible fluid.

But Euler had already in his work on gunnery [79] from 1745 noticed a strange feature of potential flow: The high pressure forming in front of a body moving through a fluid appeared to be balanced by an equally high pressure in the backi, in the case of a boat moving through water expressed as

- *...the boat would be slowed down at the prow as much as it would be pushed at the poop...*

D'Alembert followed up in 1752 by showing that potential flow in fact has zero drag, in the simplest case of a sphere by using the symmetry of potential flow with the high pressure in front balanced by an equally high pressure in the back, and the low pressure on top and bottom also balancing to give zero lift. Mathematical prediction of zero drag of e.g. a cannon ball was laughable and the Prize Problem was left unsolved and without award.

The first real test of the applicability of Calculus in fluid mechanics was disappointing, which discredited mathematical fluid mechanics among engineers from start.

In the words of the chemistry Nobel Laureate Sir Cyril Hinshelwood this resulted in an unfortunate split between the field of *hydraulics*, observing phenomena which could not be explained, and *theoretical fluid mechanics* explaining phenomena which could not be observed. A split like that of course is catstrophical from scientific point of view, but we shall see that it reamains into our days.

Chapter 4

D'Alembert's Paradox

> How wonderful that we have met with a paradox. Now we have some hope of making progress. (Nils Bohr)

> It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers to elucidate. (d'Alembert)

4.1 Zero Drag of Potential Flow

D'Alembert's Paradox formulated in 1752 states that both drag and lift of potential flow are zero: In particular a solid body should be able to move through both air and water without resistance to motion, in glaring contradiction to experience with the drag increasing roughly quadratically with the speed. Further, flight without positive lift would defy Newton's law of gravitation pulling everything to the ground. To explain flight, d'Alembert's paradox had to be resolved.

D'Alembert left his paradox to "future Geometers": Evidently something was wrong with the potential solution as an approximate solution to the Navier-Stokes equations.

4.2 What's Wrong with the Potential Solution?

Since the Navier-Stokes equations cannot be questioned as expressing Newton's 2nd law and incompressibility, there are only two possibilities:

1. the potential solution is not a correct approximate solution of the Navier-Stokes equations in the fluid domain,
2. the potential solution does not satisfy the correct boundary condition between fluid and solid.

The mathematicians of the 18th century could not resolve the paradox, neither could the 19th century masters including Navier and Stokes. When powered flight showed to be possible at the turn to the 20th century the paradox simply had to be resolved one way or the other. The emerging modern man preparing to take control over Nature, required rational science and mathematics and paradoxes and contradictions could no longer be tolerated.

4.3 Prandtl's Resolution in 1904

It was the young physicist Ludwig Prandtl who in an 8-page article in 1904 gave birth to modern fluid mechanics suggesting that the unphysical feature of the potential solution was 2: The potential solution satisfies a *slip* boundary condition allowing fluid particles to glide along the boundary without friction, while in a real fluid with positive viscosity even very small, fluid particles would have to stick to the boundary and thus form a thin *boundary layer* where the fluid speed would increase rapidly from zero at the solid boundary to the free stream value in the fluid. In other words, real physical flow would satisfy a *no-slip* condition with zero velocity on the boundary, which discriminated the potential solution as non-physical and resolved the paradox.

4.4 The Mantra of Modern Fluid Mechanics

This became the mantra of modern fluid mechanics: Both lift and drag originate from thin boundary layers casued by a no-slip boundary condition. The mantra saved theoretical fluid mechanics through the first half of the 20th century, but instead killed computational fluid dynamics emerging with the computer by asking for resolution of very thin boundary layers using quadrillions of mesh points.

4.5 New Resolution 2008

In 2008 we published a resolution of d'Alembert's paradox [104], 256 years after its formulation, which identified instead 1 as the unphysical feature: A potential solution is unstable as an approximate solution of the Navier-Stokes equations with a slip boundary condition, and because it is unstable it develops into a turbulent solution with non-zero lift and drag.

We identified the primary instability which gives the main features of the turbulent solution with lift and drag. We thus observed and explained the emergence of lift and drag of a wing and showed both to be accuractely computable by solving the NavierStokes equations with a slip boundary condition without boundary layers using some hundred thousand mesh points.

We thus computed and explained lift and drag without resorting to any effects of boundary layers and thus in particular concluded that Prandtl's mantra of modern fluid mechanics has nothing to do with reality, The consequences are far-reaching.

4.6 Suppression of Birkhoff's Innocent Question

The mathematician Garret Birkhoff at Harvard asked in his *Hydrodynamics* first published in 1950 the seemingly innocent question if there is any time-independent inviscid flow, including potential flow, which is stable, but was met with so harsh criticism by J. Stoker at Courant Institute that Birkhoff deleted the question in the second edition of his book and never wrote anything more on hydrodynamics.

But Birkhoff's question was very motivated and the active suppression delayed the progress of computational fluid dynamics by several decades. Suppression of correct science is often more harmful than the promotion of incorrect science. Unfortunately, Birkhoff did not live to see that after all he was right and Stoker wrong.

Part II

The Story of Flight

Chapter 5

The Mystery of Flight

> Few physical principles have ever been explained as poorly as the mechanism of lift [90].

5.1 What Keeps Airplanes in the Air?

The mystery of the flight of birds must have captured already the imagination of the early cave man, but it has remained a mystery from scientific point of view into our days of airborne mass transportation. How can that be? Isn't it known what keeps an airplane in the air? Is there no answer to give a curious child or someone with fear of flying? No, not if we seek a real answer [91]:

- *How do airplane wings really work? Amazingly enough, this question is still argued in many places, from elementary school classrooms all the way up to major pilot schools, and even in the engineering departments of major aircraft companies. This is unexpected, since we would assume that aircraft physics was completely explored early this century. Obviously the answers must be spelled out in detail in numerous old dusty aerodynamics texts. However, this is not quite the case. Those old texts contain the details of the math, but it's the interpretation of the math that causes the controversy. There is an ongoing Religious War over both the way we should understand the functioning of wings, and over the way we should explain them in children's textbooks.*

Of course you don't expect birds to understand why they can fly. You probably believe that somehow evolution has designed birds so that they can fly, presumably by some trial and error process over millions of year, while keeping human beings on the ground despite many efforts until Orwille and Wilbur Wright in 1903 showed that powered human flight in fact is possible by getting their *Flyer* into sustained flight with the help of a 12 horse-power engine.

But like the birds, Orwille and Wilbur did not have to really understand *why* what they managed to do was possible; they just managed somehow to do it by trial and error, building on the careful studies of bird wings by Lilienthal from the 1890s.

Orwille and Wilbur had computed that with a lift to drag ratio of 10 and a velocity of 10 meter per second, 4 effective horse-powers would get the 300 kg of the *Flyer* off ground at an effective thrust of 30 kp, and it worked! A human being of 75 kg capabable of delivering 1 horse power should also be able to take off...

Science is about understanding and understanding flight is to be able to explain from the principles of fluid mechanics how a wing can generate large lift at small drag, that is, to explain the miracle of Fig. [**?**].

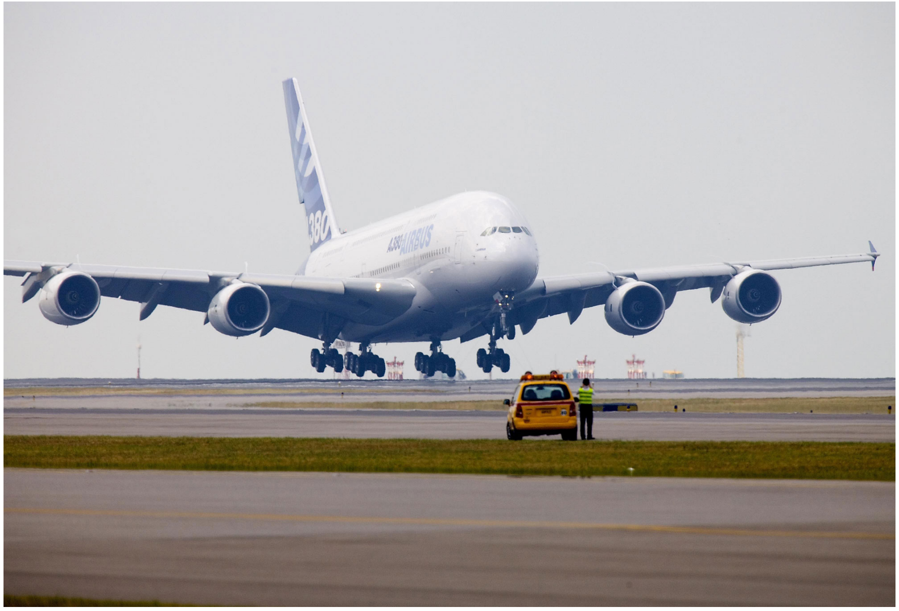


Figure 5.1: How can a 400 ton Airbus380 take off at an engine thrust of 40 tons?

5.2 Understand a Miracle

The challenge from a scientific point of view is to explain using mathematics how a fixed wing by moving horisontally through air can generate a large lift L balancing gravitation while the drag D as the horisontal force on the wing from the air is small. The lift to drag quotient L/D is typically of size $10-20$ in the *gliding flight* of birds and of airplanes at subsonic speeds, and can reach 70 for an extreme glider, which can glide 70 meters upon loosing 1 meter in altitude.

Charles Lindberg crossed the Atlantic in 1927 at a speed of 50 m/s in his 2000 kg *Spirit of St Louis* at an effective engine thrust of 150 kp (with $\frac{L}{D} = 2000/150 \approx 13$) from 100 horse powers.

By Newton's 3rd law, lift must be accompanied by *downwash* with the wing redirecting air downwards. The enigma of flight is the mechanism of a wing generating substantial downwash at small drag, which is also the enigma of sailing against the wind with both sail and keel acting like wings creating substantial lift.

Flying on wings and sailing against the wind is a miracle, and the challenge from scientific point of view is explain the miracle of $L/D > 10$. NASA pretends to explain the miracle of the flight of the *Flyer* in Fig. 5.2 as a conseqence of Newton's Third Law. Do you get it?

Flying on a barn door tilted at 45 degrees with $L/D \approx 1$ would not be a scientific miracle, but only a rocket can generate a thrust equal to its own weight and then only for a short period of time, and air transportation by rockets is nothing for birds and ordinary people, only for astronauts.

The lift force L increases quadratically with the speed and linearly with the *angle of attack*, that is the tilting of the wing from the direction of flight, until *stall* at about 15 degrees, when the drag abruptly increases and L/D becomes too small for sustained flight.

5.3 Aerodynamics as Navier-Stokes Solutions

Fluid mechanics is well described by the Navier-Stokes equations expressing conservation of mass, momentum and energy, but there are two basic issues to handle: The viscosity of the fluid is needed as input and the equations in general have turbulent solutions defying analytical description and thus have to be solved computationally using computers.

In aerodynamics or fluid dynamics of air with small viscosity, these issues come together in a fortunate way: The precise value of the small viscosity shows

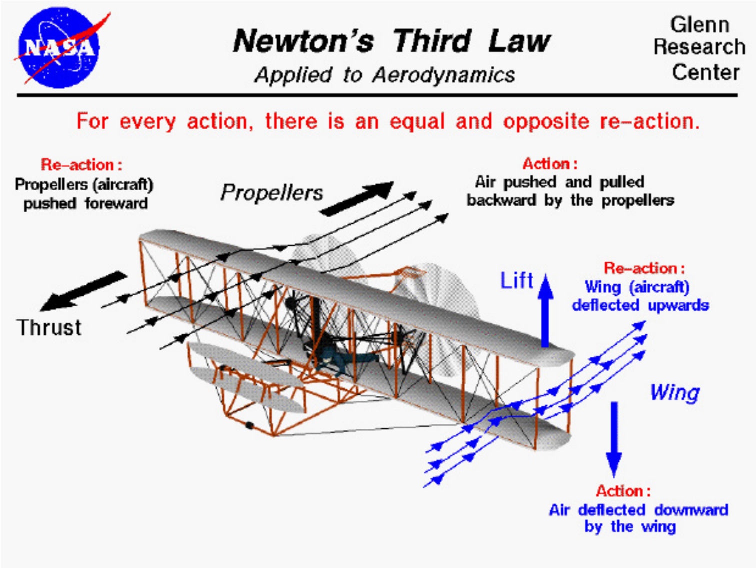

Newton's Third Law
Applied to Aerodynamics
Glenn Research Center
For every action, there is an equal and opposite re-action.
Re-action: Propellers (aircraft) pushed foreward
Propellers
Action: Air pushed and pulled backward by the propellers
Thrust
Lift
Re-action: Wing (aircraft) deflected upwards
Wing
Action: Air deflected downward by the wing

Figure 5.2: Tautological explanation of the flight of *The Flyer* by NASA:: There is upward lift on the wing from the air as a reaction to a downward push on the air from the wing.

to be largely irrelevant as concerns macroscopic quantities such as lift and drag, and the turbulent solutions always appearing in slightly viscous flow, show to be computable.

We shall find that there is a catch here, which we will adress shortly, but fortunately a catch which can be overcome, because in Einstein's words:

- *Subtle is the Lord, but malicious He is not.*

This means that the secret of flight can be uncovered by solving the Navier-Stokes equations and analyzing the computed solutions. We thus have at our disposal a complete fluid mechanics laboratory where we can study every aspect of the flow around a wing, or an entire airplane in the critical dynamics of take-off and landing, and also the flapping flight of bird. Nature can hide its secrets in real analog form, but not in digital simulation.

This leads to the program of this book:

- Solve the Navier-Stokes equations computationally.

- Study the computed turbulent solutions.

- Discover the mechanism creating large lift at small drag.
- Formalize the discovery into a new understandable theory of flight.

5.4 Computation vs Experiments

In Fig. 5.3 we show computed lift and drag (coefficients) C_L and C_D of a long NACA012 wing under varying angles of attack from cruising over take-off/landing to stall, obtained by solving the Navier-Stokes equations using automatically adapted meshes with less than 10^6 mesh points (blue curve) compared to measured values in wind tunnel experiments. The computational values lie within the range of the experimental values and thus evidently capture reality.

The message is that it is possible to compute the lift and drag of an airplane in the whole range of angles of attack, from small angles of stationary cruising at high speed, to large angles close to stall in the dynamics of start and landing at low speed. This is a happy new message, since state-of-the-art tells [59] that 50 years of doubled computer power every 18 months are need to make computation of lift and drag of an airplane possible by solving the Navier-Stokes equations.

We hope this gives the reader motivation to continue reading to discover how a wing generates large lift at small drag, the mystery of flight.

5.5 Many Old Theories: None Correct

Before revealing the essence of the new theory we address some of the many incorrect explanations of flight that you find in text books and media. If many different theories of a certain physical phenomenon coexist over time, which has been the case as concerns flight over a period of 100 years, it is a sign that none of the theories is correct. Right?

If there is a correct theory none of the incorrect theories can survive, as illustrated e.g. by the homogeneity of the (correct) homo sapiens without any surviving (incorrect) Neanderthalers.

To properly understand the correct theory, it is useful to study (some of the) incorrect theories, because even an incorrect theories usually contains an element of truth, and the correct theory is the one that best captures most elements of truth.

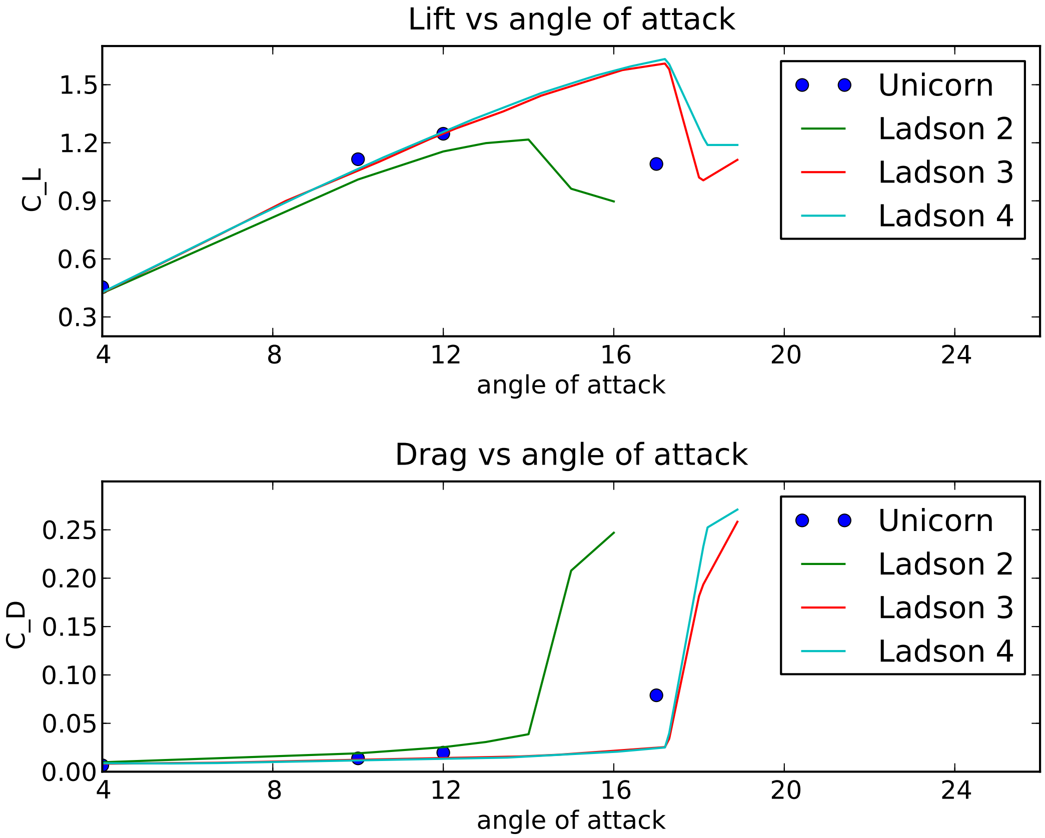


Figure 5.3: Evidence that computation of lift and drag coefficients C_L and C_D of a wing is possible from small angles of attack at crusing to large angles at start and landing: The blue curve shows computed coefficients by solving the Navier-Stokes equations by Unicorn [**?**] compared to different wind tunnel experiments by Gregory/O'Reilly and Ladson.

5.6 Correct New Theory: d'Alembert's Paradox

The new flight theory to be presented can be shown to be correct as far as the Navier-Stokes equations describe fluid mechanics, because the new flight theory directly reflects properties of Navier-Stokes solutions. The key to uncover the mathematical secret of flight came from a correct resolution of d'Alembert's paradox form 1752 presented in our book [103] and article [104] 255 years later and then with the help of computation.

Our resolution of d'Alembert's paradox is fundamentally different from the one suggested by the young German physicist Ludwig Prandtl, called the father of modern fluid mechanics, timely presented after the *Flyer* had been seen flying to form the basis of 20th century theory of flight.

By studying computed Navier-Stokes solutions, which Prandtl could not do, we discover that Prandtl's resolution of d'Alembert's paradox does not correctly describe the essential physics, nor does the flight theory conceived by the father of modern fluid mechanics. A study of the new flight theory thus naturally starts with a study of the new resolution of d'Alembert's paradox.

Before proceeding to the study of the new flight theory we will to get perspective recall how flight is explained in standard scientific literature and popular science. We start out with some popular theories listed as incorrect by NASA and complete the picture with a theory due to Kutta-Zhukovsky-Prandtl viewed to offer a scientific explanation of both lift and drag, but as we will see is also incorrect.

Chapter 6

Incorrect Theories for Uneducated

> It's all one interconnected system. Unless the overall result of that system is for air to end up lower than it was before the plane flew by, there will be no lift. Wings move air downward, and react by being pushed upward. That's what makes lift. All the rest is just interesting details[90].

> The field of hydrodynamic phenomena which can be explored with exact analysis is more and more increasing. (Zhukovsky, 1911)

6.1 Incorrect Theories: NASA

You have probably heared some of the explanations offered in popular science, like higher velocity and lower pressure on the upper surface of the wing because it is curved and air there has a longer path to travel than below? Or maybe you are an aeroplane engineer or pilot and know very well why an airplane can fly, because lif is generated by circulation?

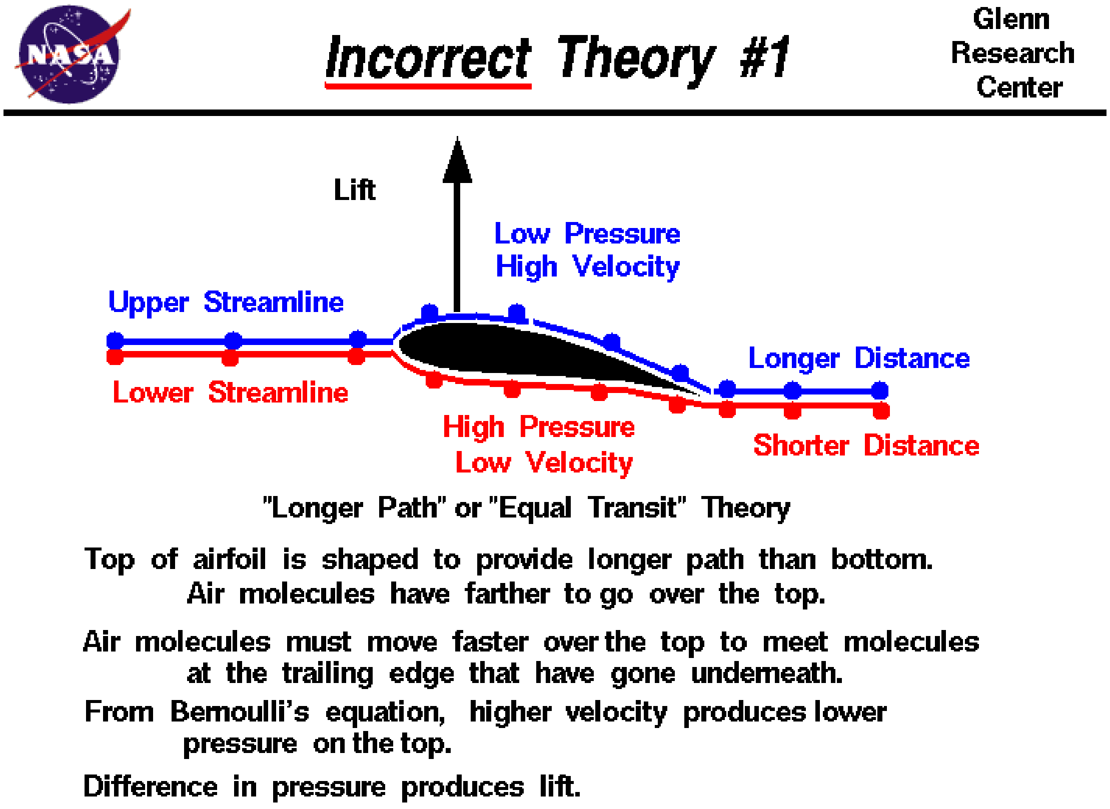
In either case, you should get a bit worried by reading that the authority NASA on its website [117] dismissses all popular science theories for lift, including your favorite one, as being incorrect, but then refrains from presenting any theory claimed to be correct! NASA surprisingly ends with an empty out of reach:

- *To truly understand the details of the generation of lift, one has to have a good working knowledge of the Euler Equations.*

This is just a fancy way of expressing that not even NASA [**?**] understands what keeps an airplane in the air. Of course, it is not possible to find the correct theory by removing all incorrect theories, like forming a correct sculpture out of a block of stone by removing all pieces of stone which are not correct, because there are infinitely many incorrect theories and not even NASA can list them all.

To present incorrect theories at length is risky pedagogics, since the student can get confused about what is correct and not, but signifies the confusion and misconceptions still surrounding the mechanisms of flight. If a correct theory was available, there would be no reason to present incorrect theories, but the absence of a correct theory is now seemingly covered up by presenting a multitude of incorrect theories.

The following three incorrect theories listed by NASA are commonly presented in text books directed to a general audience. Take a look and check out which you have met and if you they are convincing to you.



Incorrect Theory #1

Glenn Research Center

"Longer Path" or "Equal Transit" Theory

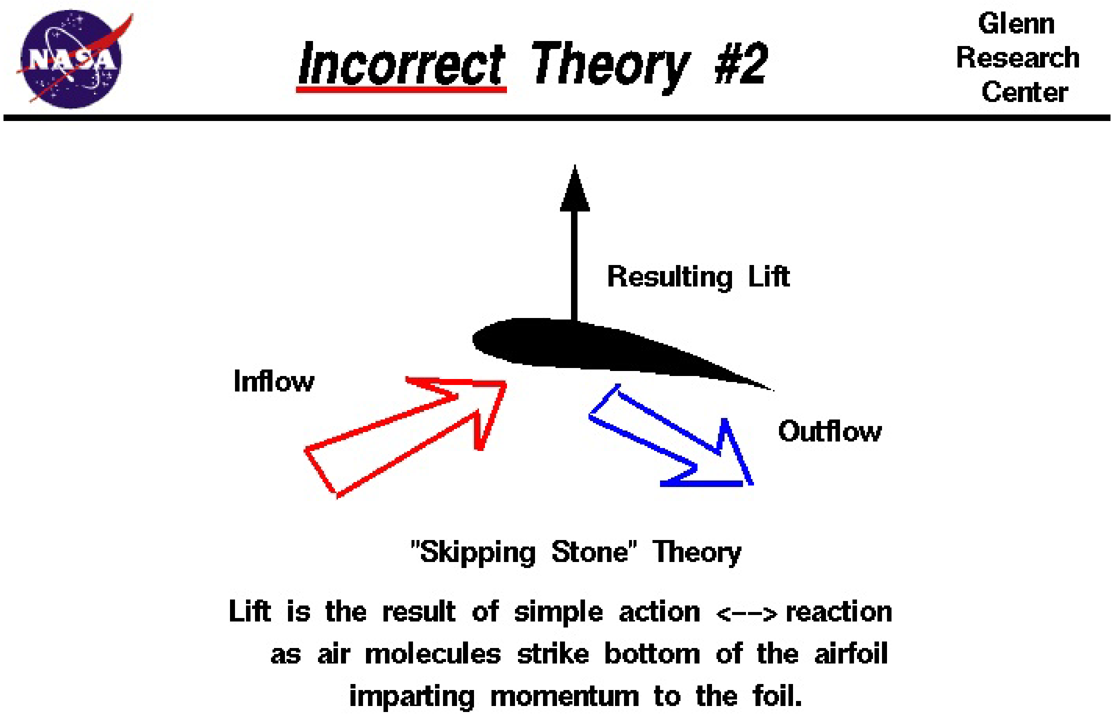
Top of airfoil is shaped to provide longer path than bottom. Air molecules have farther to go over the top.

Air molecules must move faster over the top to meet molecules at the trailing edge that have gone underneath.

From Bernoulli's equation, higher velocity produces lower pressure on the top.

Difference in pressure produces lift.

Figure 6.1: The ‘longer path theory is wrong becasue an airplane can fly upside down.



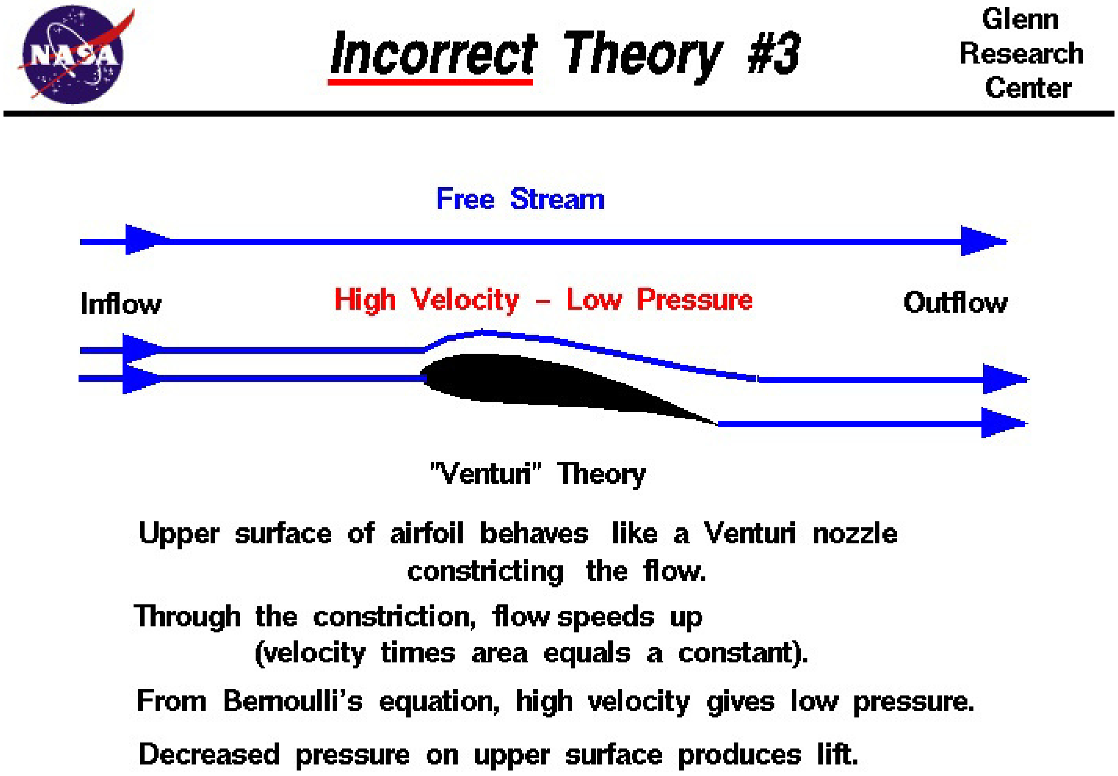
Incorrect Theory #2

Glenn
Research
Center

"Skipping Stone" Theory

Lift is the result of simple action <--> reaction
as air molecules strike bottom of the airfoil
imparting momentum to the foil.

Figure 6.2: The “skipping stone is Newton’s theory, which gives a way too small lift.



Incorrect Theory #3

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"Venturi" Theory

Upper surface of airfoil behaves like a Venturi nozzle constricting the flow.

Through the constriction, flow speeds up (velocity times area equals a constant).

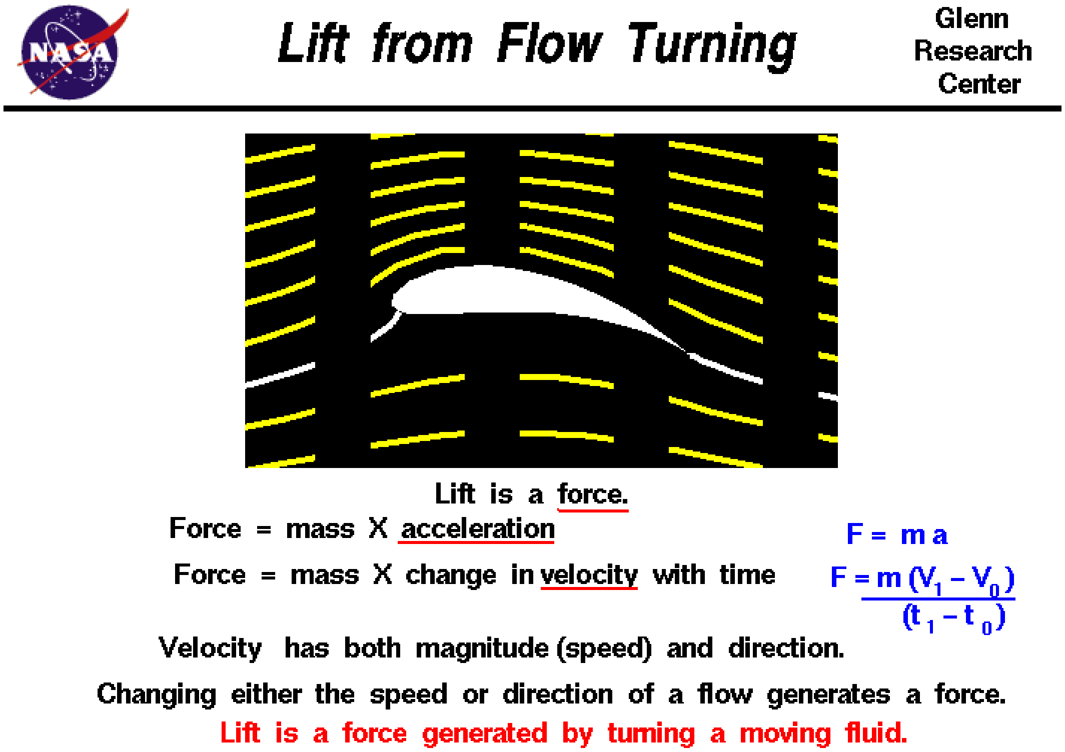
From Bernoulli's equation, high velocity gives low pressure.

Decreased pressure on upper surface produces lift.

Figure 6.3: The "Bernouilli theory of higher speed above the wing and thus lower pressure because of the curvature of the wing, is wrong because an airplane can fly upside down.

6.2 Trivial Theory: NASA

The closet NASA comes to a correct theory is the trivial theory depicted in Fig. 6.4: If there is downwash then there is lift. This follows directly by Netwon's 3rd law, but the question is why there is downwash?



Lift from Flow Turning

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Research
Center

Lift is a force.

Force = mass X acceleration $F = m\,a$

Force = mass X change in velocity with time $F = m\,\frac{(V_1 - V_0)}{(t_1 - t_0)}$

Velocity has both magnitude (speed) and direction.

Changing either the speed or direction of a flow generates a force.

Lift is a force generated by turning a moving fluid.

Figure 6.4: Trivial tautological theory of lift presented as correct by NASA.

Chapter 7

Incorrect Theory for Educated

> Lift is a lot trickier. In fact it is very controversial and often poorly explained and, in many textbooks, flat wrong. I know, because some readers informed me that the original version of this story was inaccurate. I've attempted to correct it after researching conflicting "expert" views on all this....If you're about fed up, rest assured that even engineers still argue over the details of how all this works and what terms to use [45].

7.1 Newton, d'Alembert and Wright

When the Wright brothers in 1903 showed that powered human flight was possible by putting a 12 hp engine on their *Flyer*, this was in direct contradiction to the accepted mathematical theory by Newton, who computed the lift from downwash of air particles hitting the lower part of the wing and found it to be so small that human flight was unthinkable.

Newton was supported by the mathematician d'Almembert who in 1755 proved that both lift and drag was zero for for *potential flow*, which seemed to describe the flow of air around a wing.

7.2 Kutta-Zhukovsky: Circulation: Lift

To save theoretical aerodynamics from complete collapse, a theory showing substantial lift had to be invented, and such a theory was delivered by Kutta in Germany and Zhukovsky in Russia 1904-6, who modified zero lift potential flow by adding a large scale *circulation* around the wing. Kutta-Zhukovsky showed that if there is circulation then there is lift, but could not explain from where the circulation came.

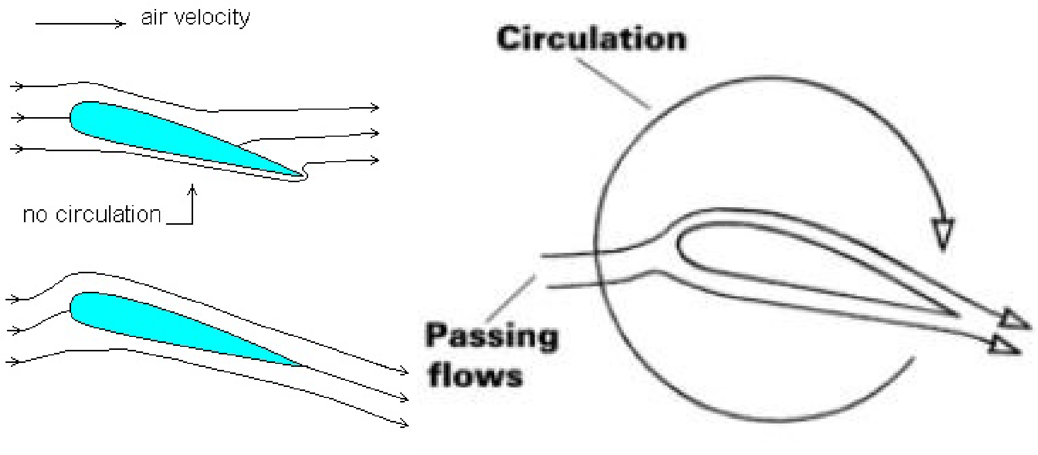


Figure 7.1: Top left figure shows potential flow without downwash (the incoming flow is not redirected) and thus no lift. The bottom left shows a flow with circulation resulting from adding a large scale rotational flow around the wing as shown in the right figure.

7.3 Prandtl: Boundary Layer: Drag King

This was timely done by the German physicists Ludwig Prandtl, who in a short note in 1904 saved aerodynamics by opening the possibility that both circulation with lift and drag somehow originiate from a thin *boundary layer*.

The theory by Kutta-Zhukovsky-Prandtl has become the theory for the educated specialists of aerodynamics, who very well understand that the theories for uneducated are incorrect. Today 100 years later, this is still the theory of flight presented in text books: Lift comes from circulation, and circulation and drag comes from a thin boundary layer.

There is an alternative to circulation as generation of lift, referred to as the *Coanda effect* stating that upper surface suction is an effect of viscosity causing the flow to stick to the surface, but the support of this theory in the literature is weak.

State-of-the-art thus tells you that to compute drag and lift of an airplane you need to resolve the boundary layer, which however requires 10 quadrillion (10^{16}

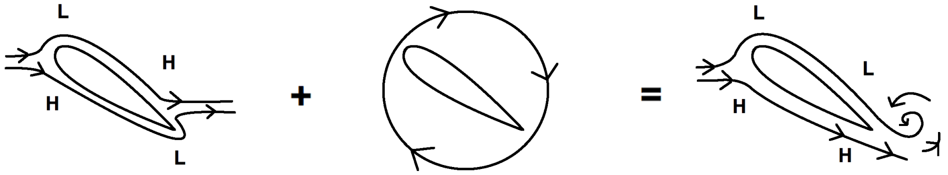


Figure 7.2: Kutta-Zhukovsky theory of lift combining potential flow (left) with large scale circulation, thus changing the zero lift pressure distribution of potential flow to lifting flow by shifting the high (H) and low (L) pressure zones at the trailing edge of the flow by unphysical circulation around the section (middle) resulting in flow with downwash/lift and starting vortex (right).

mesh points, which will require 50 years of Moore's law improving the computational power by a factor of 10^{10}.

But is the theory of Kutta-Zhukovsky-Prandtl for educated correct? Do we have to wait 50 years to compute lift and drag of an Airbus?

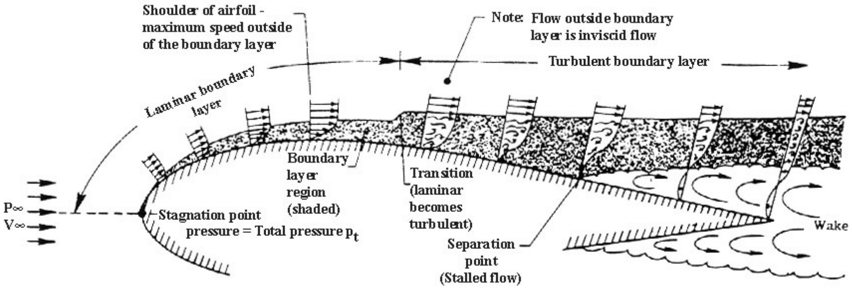

Shoulder of airfoil - maximum speed outside of the boundary layer
Note: Flow outside boundary layer is inviscid flow
Turbulent boundary layer
Laminar boundary layer
Boundary layer region (shaded)
Transition (laminar becomes turbulent)
P∞
V∞
Stagnation point pressure = Total pressure p_t
Separation point (Stalled flow)
Wake

Figure 7.3: Prandtl's boundary layer theory in pictures.

7.4 More Confusion

Many sources, in addition to NASA, give witness of the lack of convincing scientific answer of how a wing can generate lift with small drag. We give here a sample starting with more from [130]:

- *"Here we are, 100 years after the Wright brothers, and there are people who give different answers to that question," said Dr. John D. Anderson Jr.,* *the curator for aerodynamics at the Smithsonian National Air and Space Museum in Washington. "Some of them get to be religious fervor."*

- *The answer, the debaters agree, is physics, and not a long rope hanging down from space. But they differ sharply over the physics, especially when explaining it to nonscientists. "There is no simple one-liner answer to this," Dr. Anderson said.*
- *The simple Newtonian explanation also glosses over some of the physics, like how does a wing divert air downward? The obvious answer – air molecules bounce off the bottom of the wing – is only partly correct.*
- *If air has to follow the wing surface, that raises one last question. If there were no attractive forces between molecules, would there be no flight? Would a wing passing through a superfluid like ultracold helium, a bizarre fluid that can flow literally without friction, produce no lift at all? That has stumped many flight experts. "I've asked that question to several people that understand superfluidity," Dr. Anderson, the retired physicist, said. "Alas! They don't understand flight."*
- *It is important to realize that, unlike in the two popular explanations described earlier (longer path and skipping stone), lift depends on significant contributions from both the top and bottom wing surfaces. While neither of these explanations is perfect, they both hold some nuggets of validity. Other explanations hold that the unequal pressure distributions cause the flow deflection, and still others state that the exact opposite is true. In either case, it is clear that this is not a subject that can be explained easily using simplified theories. Likewise, predicting the amount of lift created by wings has been an equally challenging task for engineers and designers in the past. In fact, for years, we have relied heavily on experimental data collected 70 to 80 years ago to aid in our initial designs of wing.*[52]
- *http* $: //www.youtube.com/watch?v = uUMlnIwo2Qo$
- *http* $: //www.youtube.com/watch?v = ooQ1F2jb10A$
- *http* $: //www.youtube.com/watch?v = kXBXtaf2TTg$
- *http* $: //www.youtube.com/watch?v = 5wIq75_BzOQ$
- *http* $: //www.youtube.com/watch?v = khca2FvGR - w$

Chapter 8

Summary of State-of-the-Art

> You'd think that after a century of powered flight we'd have this lift thing figured out. Unfortunately, it's not as clear as we'd like. A lot of half-baked theories attempt to explain why airplanes fly. All try to take the mysterious world of aerodynamics and distill it into something comprehensible to the lay audience–not an easy task. Nearly all of the common ”theories” are misleading at best, and usually flat-out wrong [87].

8.1 Newton

Classical mathematical mechanics could not give an answer to the mystery of gliding flight: Newton computed by elementary mechanics the lift of a tilted flat plate redirecting a horisontal stream of fluid particles, but obtained a disappointingly small value proportional to the square of the angle of attack. To Newton the flight of birds was inexplicable, and human flight certainly impossible.

8.2 d'Alembert and Potential Flow

D'Alembert followed up in 1752 by formulating his paradox about zero lift/drag of *inviscid incompressible irrotational steady flow* referred to as *potential flow,* which seemed to describe the airflow around a wing since the viscosity of air is very small so that it can be viewed as being inviscid (with zero viscosity). Mathematically, potential flow is given as the gradient of a *harmonic funtion* satisfying *Laplace's equation*.

At speeds less than say 300 km/h air flow is almost incompressible, and since a wing moves into still air the flow it could be be expected to be irrotational without swirling rotating vortices. D'Alembert's mathematical potential flow thus seemed to capture physics, but nevertheless had neither lift nor drag, against all physical experience. The wonderful mathematics of potential flow and harmonic functions thus showed to be without physical relevance: This is *D'Alembert's paradox* which came to discredit mathematical fluid mechanics from start [104, 125, 92].

To explain flight d'Alembert's paradox had to be resolved, but nobody could figure out how and it was still an open problem when Orwille and Wilbur Wright in 1903 showed that heavier-than-air human flight in fact was possible in practice, even if mathematically it was impossible.

8.3 Kutta-Zhukovsky-Prandtl

Mathematical fluid mechanics was then saved from complete collapse by the young mathematicians Kutta and Zhukovsky, called the father of Russian aviation, who explained lift as a result of perturbing potential flow by a large-scale circulating flow or *circulation* around the two-dimensional section of a wing, and by the young physicist Prandtl, called the father of modern fluid dynamics, who explained drag as a result of a *viscous boundary layer* [119, 120, 124, 93].

This is the basis of state-of-the-art [21, 72, 74, **?**, 98, 126, 137], which essentially is a simplistic theory for lift without drag at small angles of attack in inviscid flow and for drag without lift in viscous flow. However, state-of-the-art does not supply a theory for lift-and-drag covering the real case of *3d slightly viscous turbulent* flow of air around a 3d wing of a jumbojet at the critical phase of take-off at large angle of attack (12 degrees) and subsonic speed (270 km/hour), as evidenced in e.g. [47, 89, 90, 91, 130, 7, 52, 113, 116]. The simplistic theory allows an aeroplane engineer to roughly compute the lift of a wing a crusing speed at a small angle of attack, but not the drag, and not lift-and-drag at the critical phase of take-off [59, 131]. The lack of mathematics has to be compensated by experiment and experience. The first take off of the new Airbus 380 must have been a thrilling experience for the design engineers.

The state-of-the-art theory of flight can be summarized as either

- correct and trivial,
- nontrivial and incorrect,

in the following forms:

- Downwash generates lift: trivial without explanation of reason for downwash from suction on upper wing surface.
- Low pressure on upper surface: trivial without explanation why.
- Low pressure on curved upper surface because of higher velocity (by Bernouilli's law), because of longer distance: incorrect.
- Coanda effect: The flow sticks to the upper surface by viscosity: incorrect.
- Kutta-Zhukovsky: Lift comes from circulation: incorrect.
- Prandtl: Drag comes mainly from viscous boundary layer: incorrect.

8.4 Text Book Theory of Flight

The following text books all present versions of the Kutta-Zhukovsky-Prandtl theory of flight:

- Prandtl-Essentials of Fluid Mechanics, Herbert Oertel (Ed.)
- Aerodynamics of Wingd and Bodies, Holt Ashley and Marten Landahl,
- Introduction to the Aerodynamics of Flight, Theodore A. Talay, Langley Reserach Center,
- Aerodynamics of the Airpoplane, Hermann Schlichting and Erich Truckenbrodt,Mac Graw Hill
- Airplane Aerodynamics and Performance, Jan Roskam and C T Lan,
- Fundamentals of Aerodynamics, John D Anderson,
- Fuhrer durch die Stromungslehre, L Prandtl,...
- Aerodynamicss of Wind Turbins, Martin Hansen,
- Aerodynamics, Aeronautics and Flight Mechanics, McCormick,
- Aerodynamics, Krasnov,

- Aerodynamics, von Karmann,
- Theory of Flight, Richard von Mises.

The text book theory of flight has been remarkably stable over 100 year with little improvement in accuracy as if the theory once and for all was set by Kutta-Zhukovsky-Prandtl. But science does not work that way: If no progress is made on a complex scientific topic, like flight, this is a strong indication that what was hammered in stone is incorrect science. This book will help the reader to decide if such a suspicion is well founded.

Chapter 9

Shut Up and Calculate

> How can aviation be grounded in such a muddy understanding of the underlying physics? As with many other scientific phenomena, it's not always necessary to understand why something works to make use of it. We engineers are happy if we've got enough practical knowledge to build flying aircraft. The rest we chalk up to magic [87].

9.1 Naviers-Stokes vs Schrödinger

The interpretation of the wave functions of the new quantum mechanics of the atomic world as solutions of Schrödinger's wave equations, was intensely debated in the 1920s by Schrödinger, Bohr, Born and Dirac without reaching any agreement. This led a frustated Dirac in an effort to get out of the scientific dead lock to make the appeal ”shut up and calculate”: Simply solve the Schrödinger equation and take what you get as physics. However, Dirac's appeal did not and still does not help much because analytical solution of Schrödinger's equation is possible only for the Hydrogen atom with one electron, and computational solution involves $3N$ space dimensions for N particles, which even today is possible only for small N.

The debate thus is shifted to different techniques of solving Schrödinger's equation, considered to harbor the truth, and the debate goes on.

Similarly, solutions of the Navier-Stokes equations may be expected to tell the physics of flight, and so the question is if they are computable or as uncomputable as Schrödinger's equation?

Navier-Stokes solutions at the high Reynold's numbers of flight are turbulent and have thin boundary layers and the standard wisdom expressed by Kim an Moin [43] is that computational resolution requires quadrillions of mesh points beyond the capacity of any forseeable computer.

But there is trick, or mircale, which make the Navier-Stokes equations computable: If we combine the Navier-Stokes equations with a slip boundary condition modeling that the friction force from the air on the wing is small, which it is for slightly viscous flow, then there is no Prandtl boundary layer to resolve and then solutions of the Naviers-Stokes equations can be computed with 10^6 mesh points, and the lift and drag of these solutions agree very well with experiments. Thus Dirac's appeal works out for flight described by Navier-Stokes equations.

Chosing a slip (or small friction) boundary condition, which we can do because it is a good model of actual physics, we gain in two essential aspects: Solutions become computable and computed solutions tell the truth.

The truth we find this way is that neither lift nor drag originate from a thin boundary layer, and thus that the Kutta-Zhukovsky-Prandtl theory is unphysical and incorrect. State-of-the-art today thus presents a theory of flight which is both unphysical and uncomputable.

9.2 Compute - Analyze - Understand

Computed solutions show that a wing creates lift as a reaction force *downwash*, with less than 1/3 coming from the lower wing surface pushing air down and the major remaining part from the upper surface sucking air down, with a resulting *lift/drag quotient* $\frac{L}{D} > 10$.

You could stop here following the device of the physicist Dirac of "shut up and calculate" but as a scientist and rational human beings you would certainly like to "understand" the solutions, that is describe the "mechanism" making a wing generate large lift with small drag.

Chapter 10

Mathematical Miracle of Flight

> ...do steady ow ever occur in nature, or have we been pursuing fantasy all along? If steady flows do occur, which ones occur? Are they stable, or will a small perturbation of the ow cause it to drift to another steady solution, or even an unsteady one? The answer to none of these questions is known. (Marvin Shinbrot in Lectures on Fluid Mechanics, 1970)

Computing turbulent solutions of the Navier-Stokes equations led us to a resolution of d'Alembert's paradox, which revealed the miracle of flight as we show in this book. You will find that it is quite easy to grasp, because it can be explained using different levels of mathematics. We start out easy with the basic principle in concept form and then indicate some of the mathematics with references to more details. Supporting information is given in the Google knols [135] and [111].

To understand flight means to identify the relevant mathematical aspects of solutions to the Navier-Stokes equations, and this is what this book will help you to do. The code words of the mathematical mircale of flight:

- Non-separation of potential flow before trailing edge creating suction on the upper surface of wing and downwash.
- Slip-separation at trailing edge with small drag instead of potential flow separation destroying lift.

This principle is pictured in Fig. 10.1 supported by computational simulations in Fig. 10.2- **??**. We see the zero lift/drag of potential flow being replaced by real flow with low-pressure trailing edge slip separation with streamwise vorticity swirling flow shown in Fig. 10.3. We shall below analyze this pictures in more detail to reach a real understanding of how a wing is capable of generating at the ame time big lift and small drag, the enigma of flight. The reader may anticipate the analysis by a study of the pictures.

The enigma of flight is why the air flow separates from the upper wing surface at the *trailing edge*, and not before, with the flow after separation being redirected downwards according to the tilting of the wing or *angle of attack*. We will reveal the secret to be an effect of a fortunate combination of features of *slightly viscous incompressible flow* including a crucial *instability mechanism at separation* analogous to that seen in the swirling flow down a bathtub drain, generating both suction on the upper wing surface and drag.

We show that this mechanism of lift and drag is operational for angles of attack smaller than a critical value of about 15 degrees depending on the shape of the wing, for which the flow separates from the upper wing surface well before the trailing edge with a sudden increase of drag and decrease of lift referred to as *stall*.

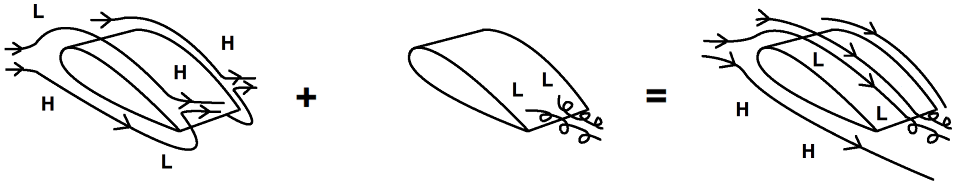


Figure 10.1: Correct explanation of lift by perturbation of potential flow (left) at separation from physical low-pressure turbulent counter-rotating rolls (middle) changing the pressure and velocity at the trailing edge into a flow with downwash and lift (right).

10.1 Mathematical Computation and Analysis

We present below a mathematical analysis of the computed solutions showing that lift and drag result from a specific 3d instability mechanism generating low-pressure rolls of streamwise vorticity attaching at separation. This analysis gives a new resolution of d'Alembert's paradox of zero drag in inviscid flow [**?**], which is fundamentally different from the accepted resolution by Prandtl based on boundary layer effects.

We show that lifting flow results from an instability at rear separation generating counter-rotating low-pressure rolls of streamwise vorticity inititated as surface vorticity resulting from meeting opposing flows. This mechanism is entirely

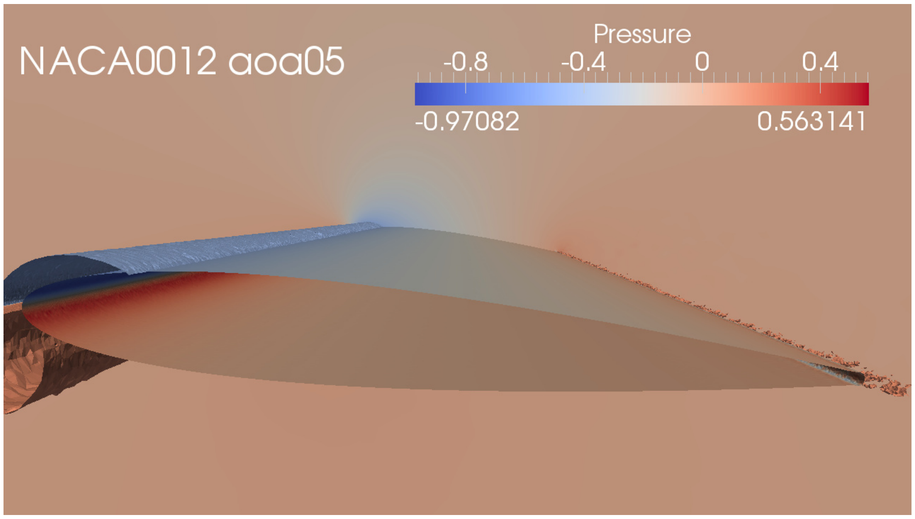


Figure 10.2: Pressure distribution with big leading edge lift (suction above and push below) and small drag at $\alpha = 5$. Notice that the high pressure at the trailing edge of potential flow is missing thus maintaining big lift.

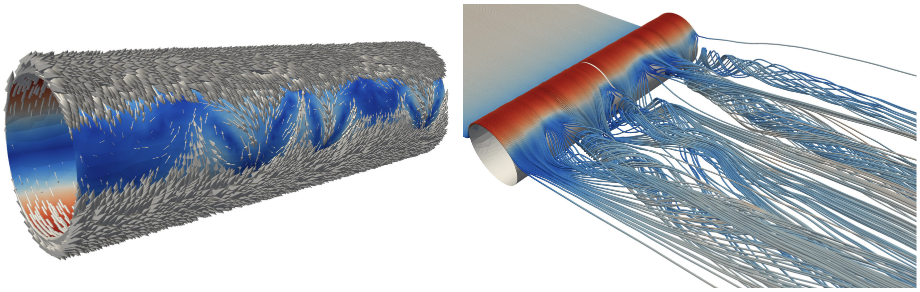


Figure 10.3: Turbulent separation by surface vorticity forming counter-rotating low-pressure rolls in flow around a circular cylinder, illustrating separation at the trailing edge of a wing [101].

different from the mechanism based on global circulation of Kutta-Zhukovsky theory. We show that the new theory allows accurate computation of lift, drag and twisting moments of an entire airplane using a few millions of mesh-points, instead of the impossible quadrillions of mesh-points required according to state-of-the-art following Prandtl's dictate of resolution of very thin boundary layers connected with no-slip velocity boundary conditions.

The new theory thus offers a way out of the present deadlock of computational aerodynamics of slightly viscous turbulent flow.

Chapter 11

Observing Pressure, Lift and drag

11.1 Potential Flow Modified at Separation

We will now uncover the secret of flight in more detail by inspecting computational solutions of Navier-Stokes equations with a slip boundary condition. The distribution of the pressure over the wing surface determines both lift and drag. The pressure acts in a direction normal to the wing surface and its components perpendicular and parallel and perpendicular to the motion of the wing give the distributions of lift and drag over the wing surface which give the total lift L and drag D with by integration. We discover that $L/D \approx 30 - 50$ for $\alpha < 14$ as recorded in Fig. 5.3, which captures the miracle of flight.

We will then inspect the velocity and the vorticity and the seek to rationalize what we have seen in terms of basic fluid mechanics. We will find that the flow can be described as a modified form of potential flow with the modification resulting from a specific form of separation at the trailing edge which we will describe as *slip separation with point stagnation*.

The flow before separation thus will seen to be close to potential flow and there conform with *Bernoulli's Principle* stating that the sum of kinetic energy $\frac{1}{2}|u|^2$ with u velocity and pressure p remains constant over the region of potential flow:

$$\frac{1}{2}|u|^2 + p = \text{constant}, \tag{11.1}$$

expressing that the pressure is low where the velocity is high and vive versa.

We recall the principal description in Fig. 10.1 with full potential flow to the left with high pressure H on top of the wing and low pressure L below at the trailing edge destroying lift, and to the right the real flow modified by a certain perturbation at the trailing edge switching H and L to generate lift.

11.2 Pressure, Lift and Drag Distribution

Fig. 11.2-11.2 show the pressure distribution for $\alpha = 4, 10, 12, 17$. We observe

- low pressure on top/front of the leading edge and high pressure below as expected from potentiual flow,
- no high pressure on top of the trailing edge in contrast to potential flow,
- the pressure distribution intensifies with increasing angle of attack,
- max negative pressure (lift) 5 times bigger than max pressure (drag) on leading edge for $\alpha = 10$,

Fig. 11.5 shows the distributions of lift and drag over the surface of the wing section for $\alpha = 0, 2, 4, 10, 18$. We observe

- lift increases and peaks at the leading edge as α increases towards stall,
- both lift and drag are small att the trailing edge,
- the negative lift on the lower surface for small α shifts to positive lift for larger α,
- the lift from the upper surface is several times bigger than from the lower surface
- leading edge suction: negative/positive drag on the upper/lower leading edge balance to give small net drag,
- main drag from leading edge for smaller α.

Altogether, we see that the miracle of $\frac{L}{D} > 10$ results from the facts that

- main lift and drag come from leading edge,
- minimal pressure several times bigger than the maximal pressure on the leading edge (factor 5 say)
- leading edge suction reduces drag (another factor 2 say).

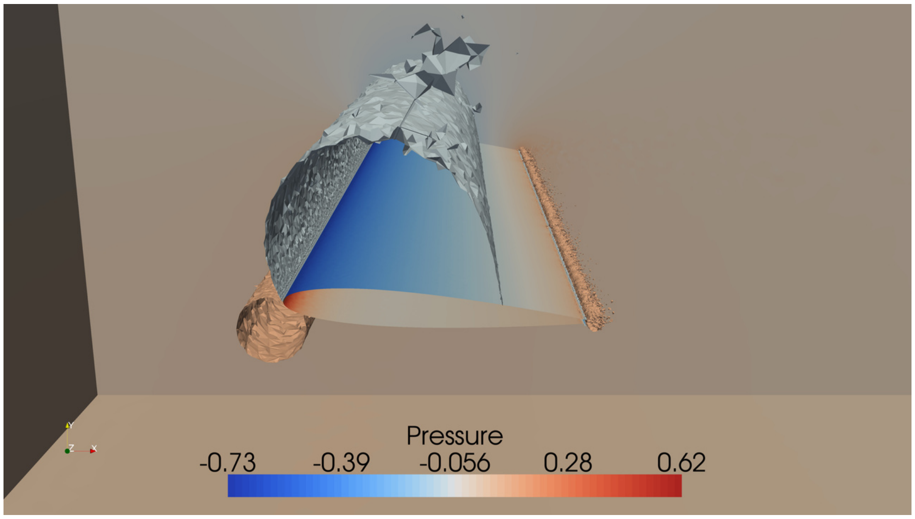


Figure 11.1: Pressure distribution at $\alpha = 4$.

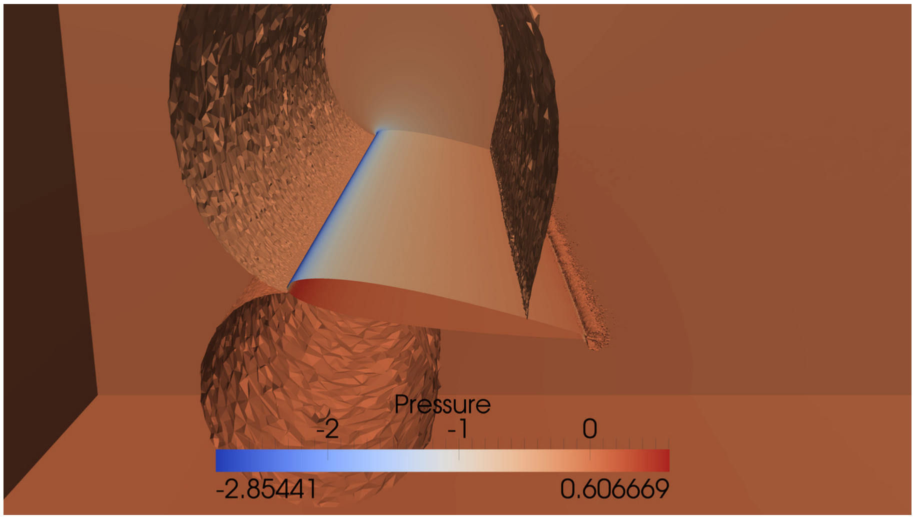


Figure 11.2: Pressure at $\alpha = 10$.

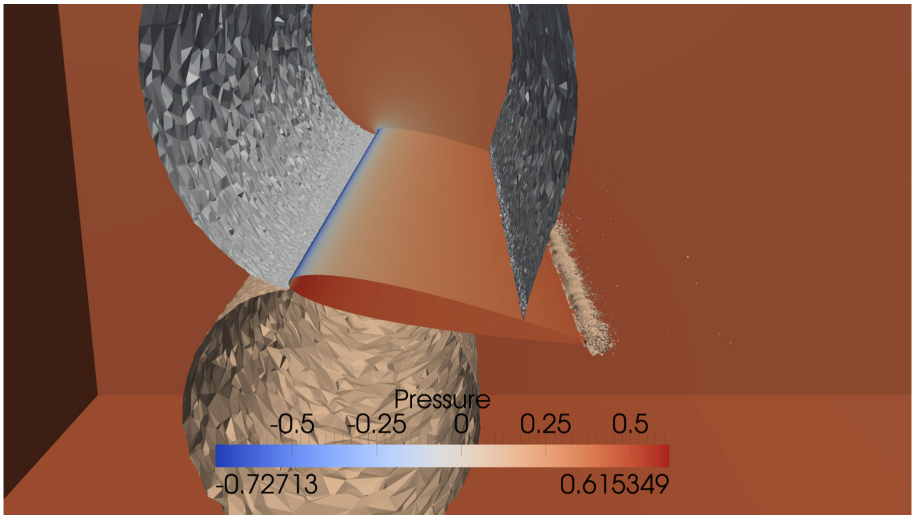


Figure 11.3: Pressure distribution at $\alpha = 12$.

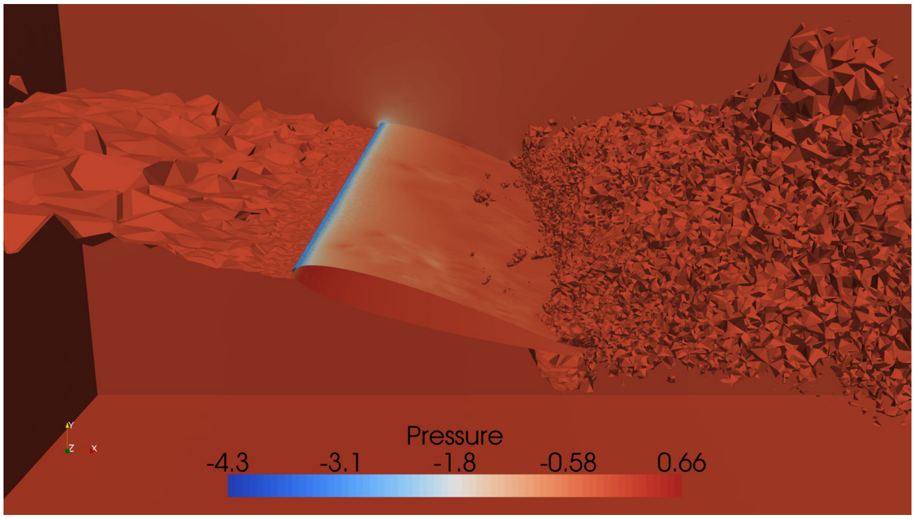


Figure 11.4: Pressure at $\alpha = 17$.

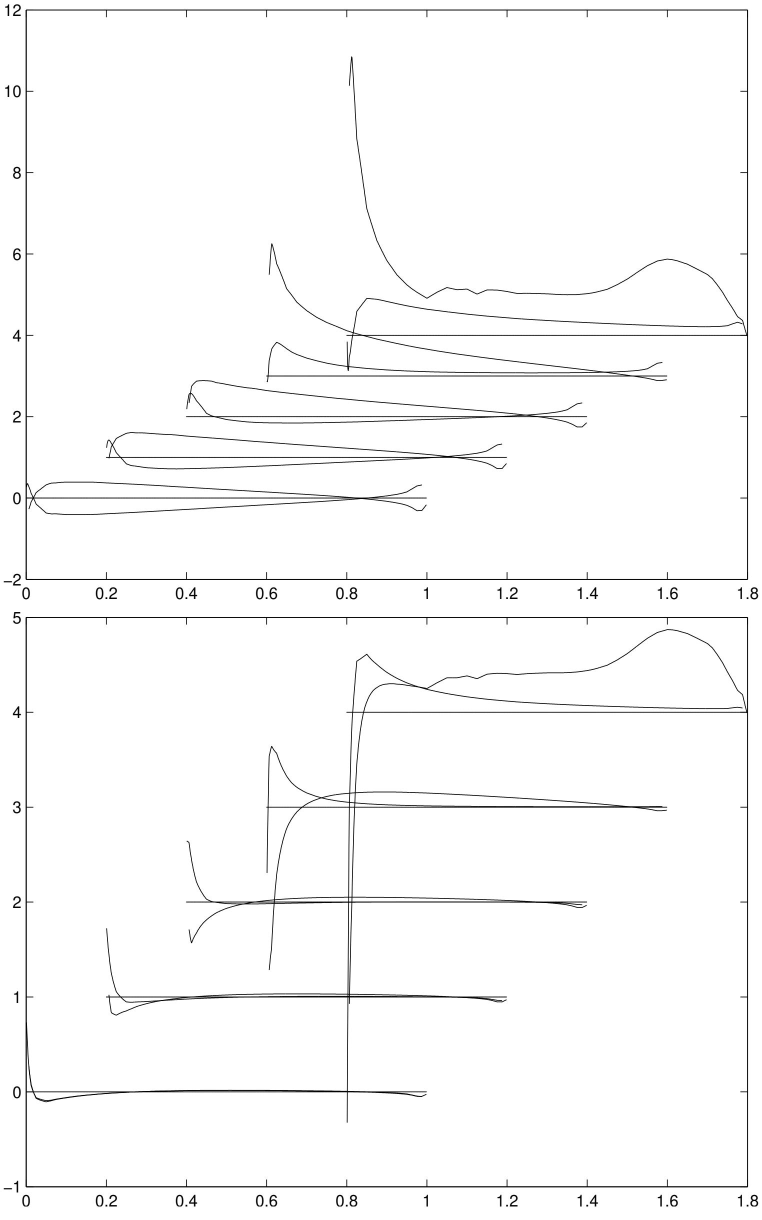


Figure 11.5: G2 computation of normalized local lift force (upper) and drag force (lower) contributions acting along the lower and upper parts of the wing, for angles of attack 0, 2 ,4 ,10 and 18°, each curve translated 0.2 to the right and 1.0 up, with the zero force level indicated for each curve.

Chapter 12

Observing Velocity

Fig. 12 and 12 show the velocity for $\alpha = 10, 14, 17$. We observe

- high velocity on top/front of the leading edge and low velocity below in conformity with Bernoulli's Principle,
- a wake with low velocity develops as stall is approached,
- well before stall the flow separates smoothly at the trailing edge and not as potential flow at stagnation before the trailing edge,
- the separation moves forward from the trailing edge on the upper surface as stall is approached.

Fig. 12.3 and 12.4 show oilfilm plots of the velocity at the trailing edge. We observe

- rotational slip separation with point stagnation in a zig-zag pattern
- generating rolls of streamwise vorticity with low pressure.

Fig. 12.6-12.10 also show the zig-zag velocity separation pattern with point stagnation.

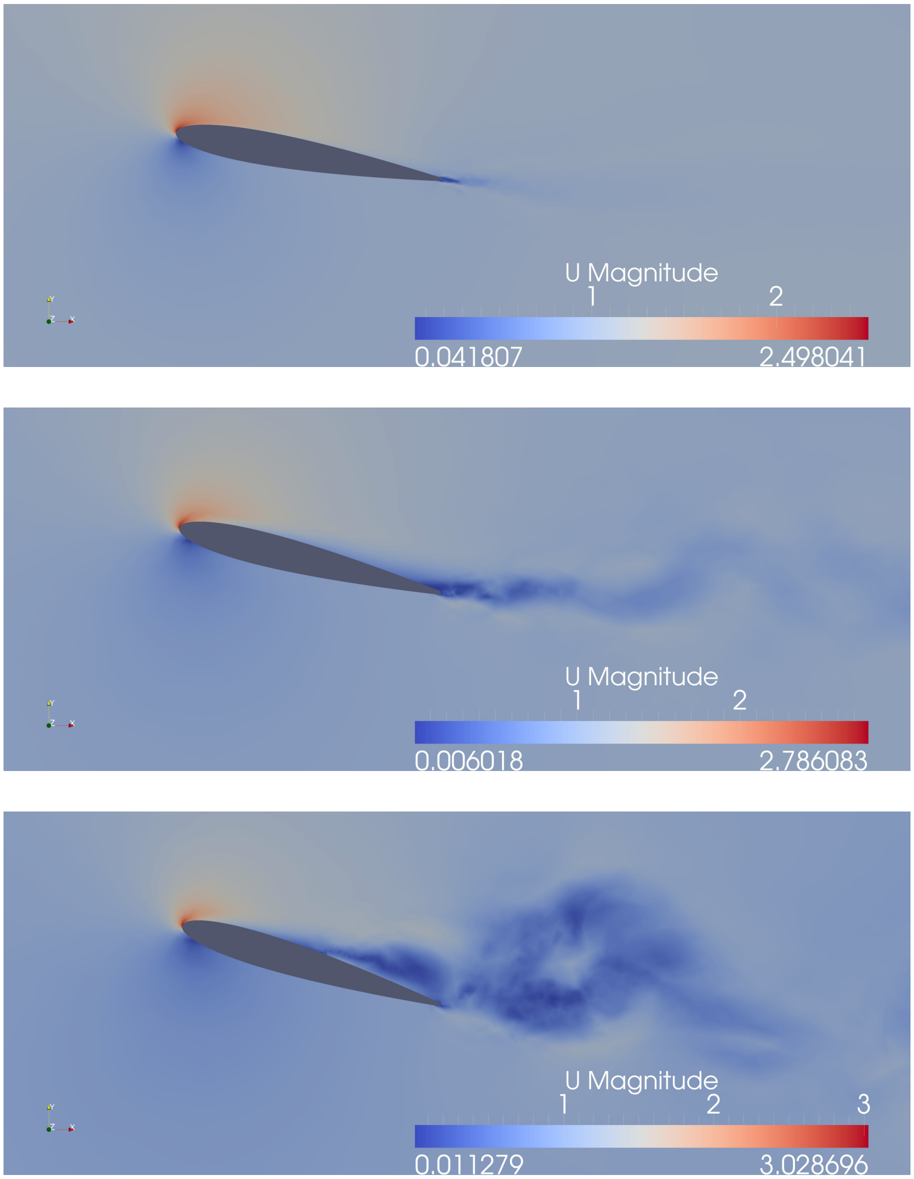


Figure 12.1: Velocity magnitude around the airfoil for $\alpha = 10$ (top), 14 (center) and 17 (bottom).

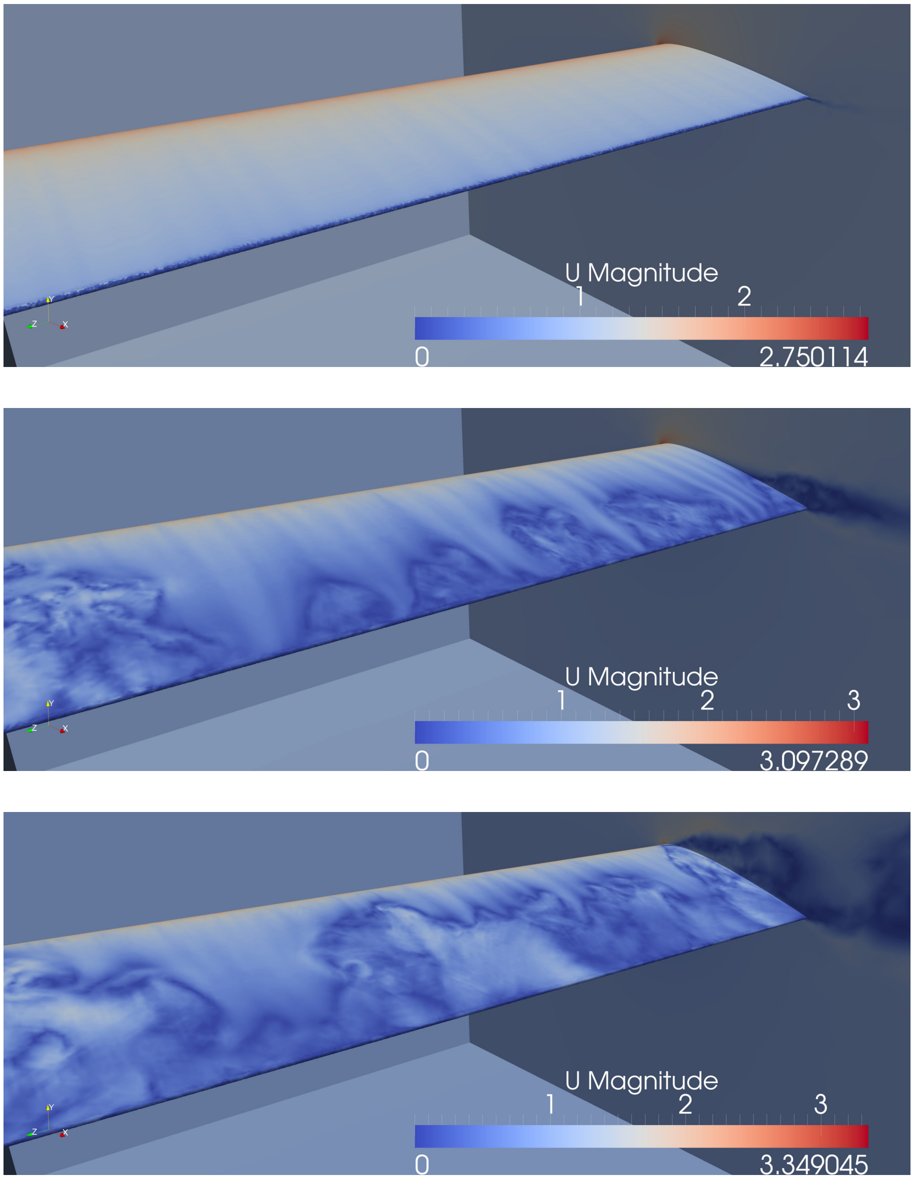


Figure 12.2: Velocity magnitude on the airfoil surface for $\alpha = 10$ (top), 14 (center) and 17 (bottom) showing that separation pattern moves up the airfoil with increasing α towards stall.

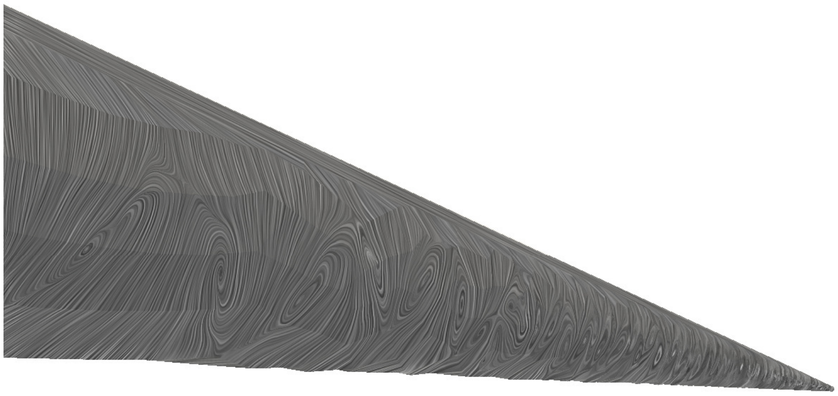


Figure 12.3: G2 oilfilm plots showing separation pattern with point stagnation for $aoa = 4$.

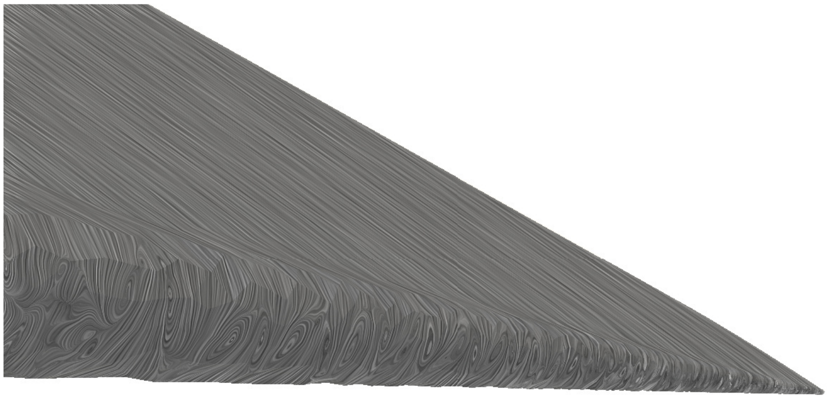


Figure 12.4: G2 oilfilm plots showing separation pattern with point stagnation $aoa = 12$.

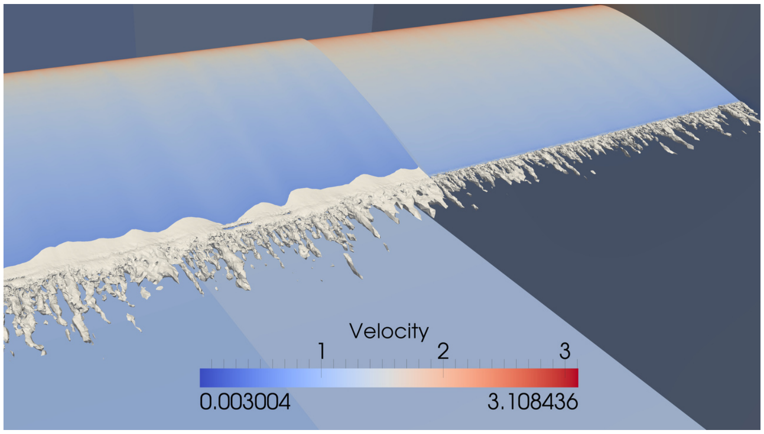


Figure 12.5: Velocity $aoa = 10$.

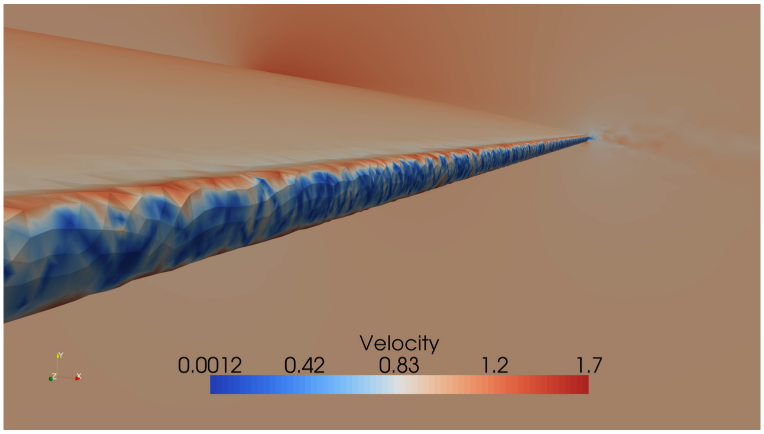


Figure 12.6: Trailing edge zig-zag velocity pattern $aoa = 04$.

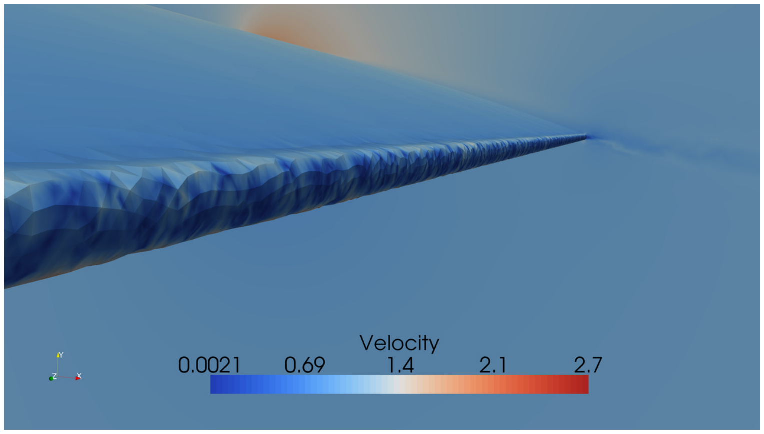


Figure 12.7: Trailing edge zig-zag velocity $aoa = 10$.

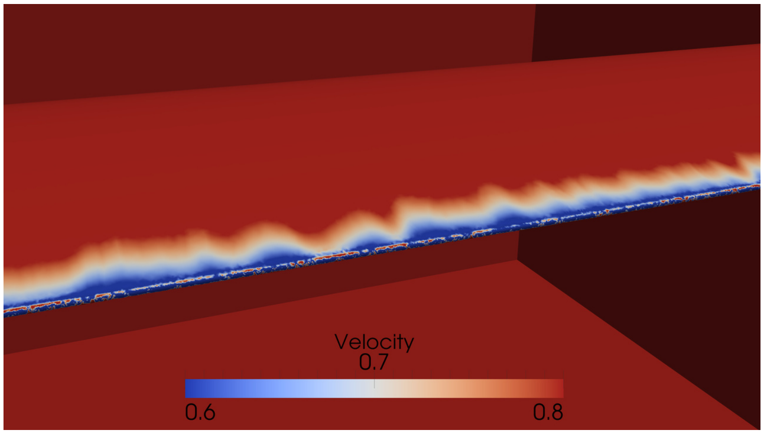


Figure 12.8: Zigzag velocity $aoa = 12$.

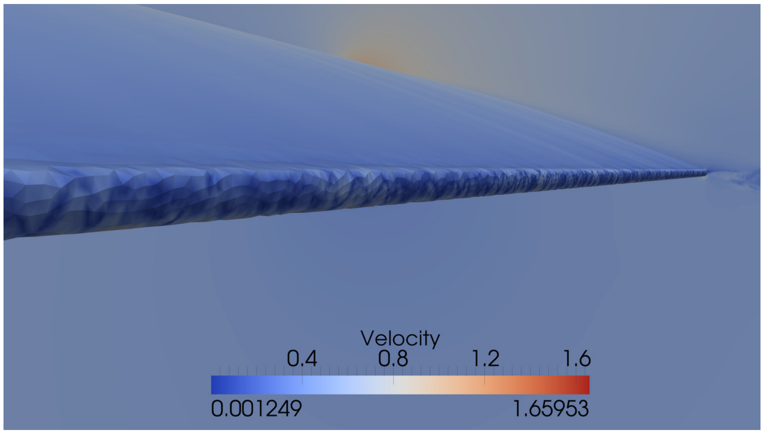


Figure 12.9: Trailing edge zig-zag velocity $aoa = 12$.

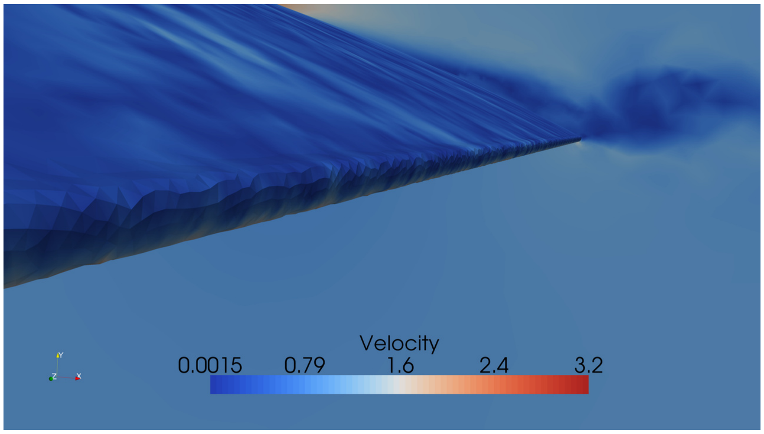


Figure 12.10: Trailing edge zig-zag velocity $aoa = 17$.

Chapter 13

Observing Vorticity

Fig. 13.1-13.3 shows side view of pressure, velocity and top view of vorticity for $\alpha = 2, 4, 8, 10, 14, 18$. We observe

- development of rolls of streamwise vorticity at separation at the trailing edge as illustrated in Fig. 10.1,
- pressure is small inside rolls of streamwise vorticity generating som drag,
- separation moves from the trailing edge to upper surface as α approches stall.

Below we will in more detail study the dynamics of the generation of streamwise vorticity as the generic structure of rotational slip separation with point stagnation.

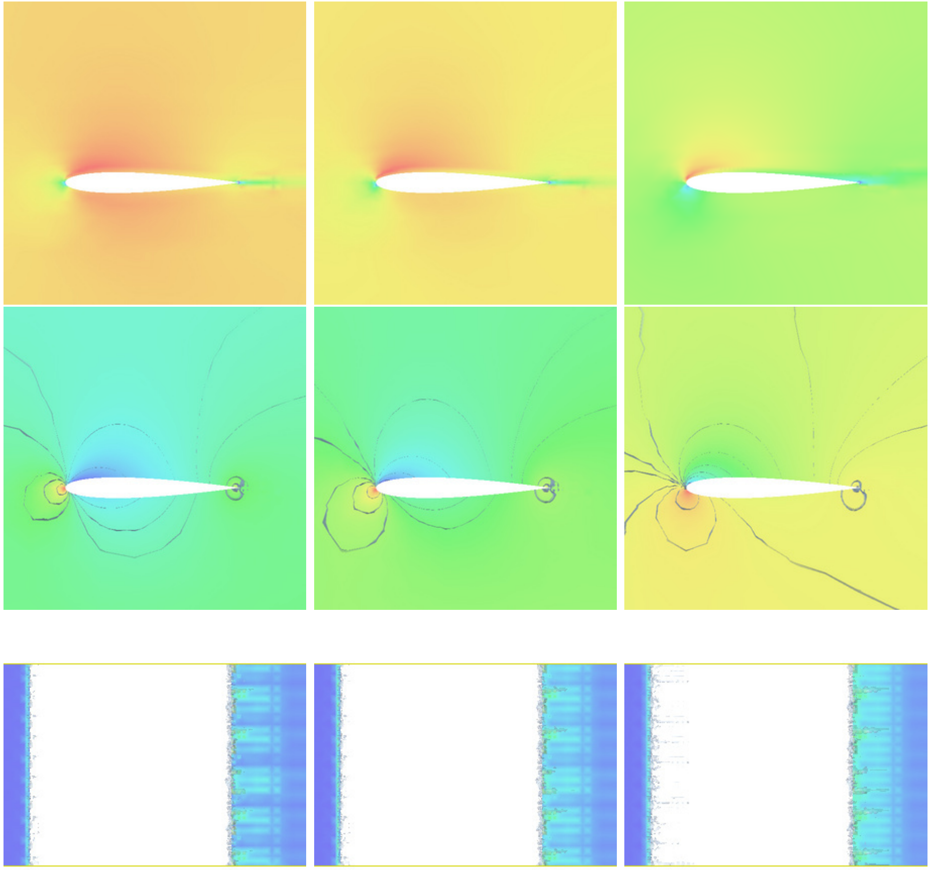


Figure 13.1: G2 computation of velocity magnitude (upper), pressure (middle), and non-transversal vorticity (lower), for angles of attack 2, 4, and 8° (from left to right). Notice in particular the rolls of streamwise vorticity at separation.

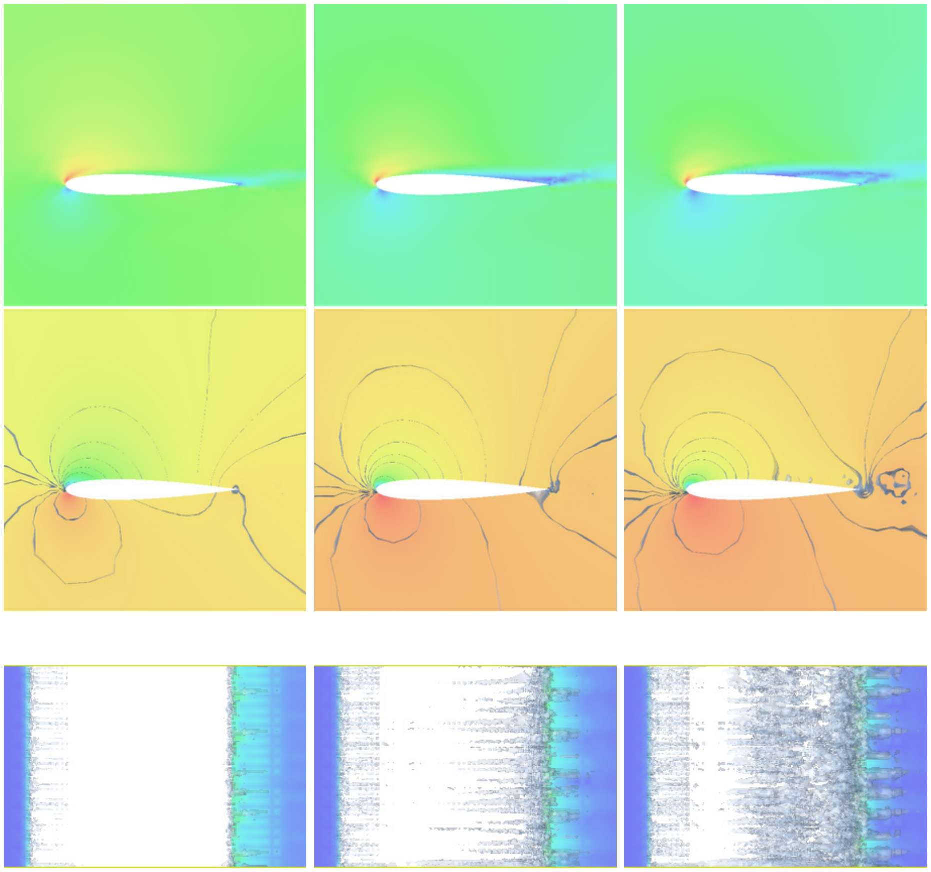


Figure 13.2: G2 computation of velocity magnitude (upper), pressure (middle), and topview of non-transversal vorticity (lower), for angles of attack 10, 14, and 18° (from left to right). Notice in particular the rolls of streamwise vorticity at separation.

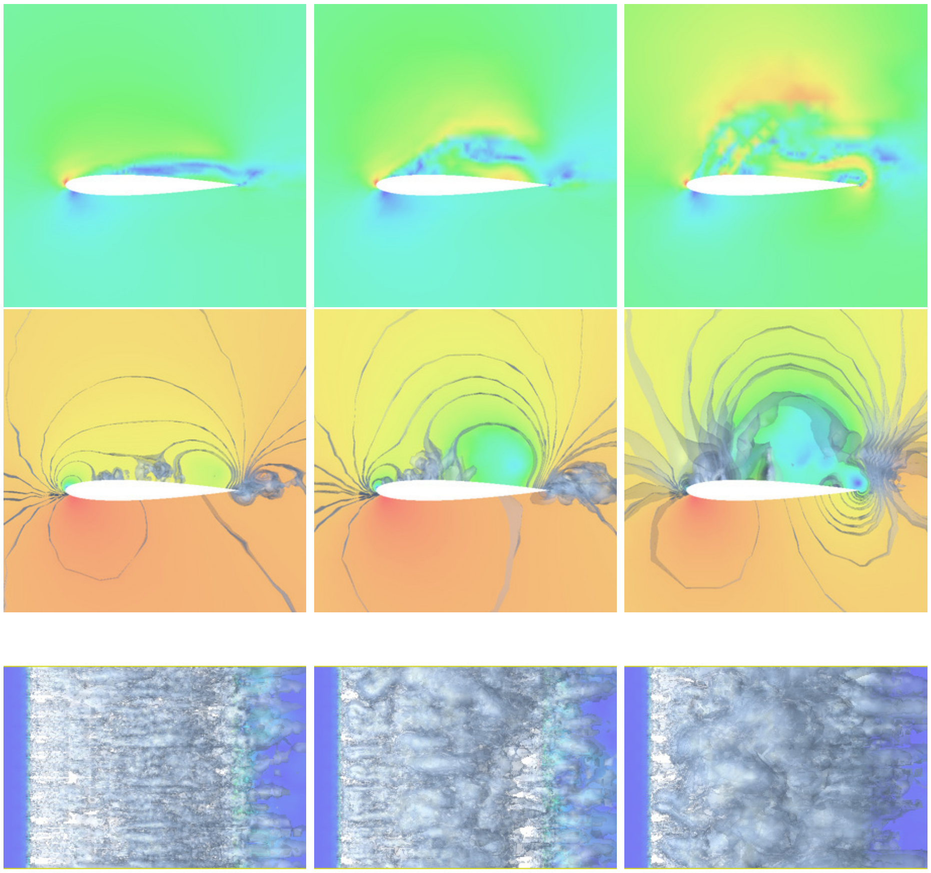


Figure 13.3: G2 computation of velocity magnitude (upper), pressure (middle), and non-transversal vorticity (lower), for angles of attack 20, 22, and 24° (from left to right).

Chapter 14

Summary of Observation

Phase 1: $0 \leq \alpha \leq 8$

At zero angle of attack with zero lift there is high pressure at the leading edge and equal low pressures on the upper and lower crests of the wing because the flow is essentially potential and thus satisfies Bernouilli's law of high/low pressure where velocity is low/high. The drag is about 0.01 and results from rolls of low-pressure streamwise vorticity attaching to the trailing edge. As α increases the low pressure below gets depleted as the incoming flow becomes parallel to the lower surface at the trailing edge for $\alpha = 6$, while the low pressure above intenisfies and moves towards the leading edge. The streamwise vortices at the trailing edge essentially stay constant in strength but gradually shift attachement towards the upper surface. The high pressure at the leading edge moves somewhat down, but contributes little to lift. Drag increases only slowly because of negative drag at the leading edge.

Phase 2: $8 \leq \alpha \leq 12$

The low pressure on top of the leading edge intensifies to create a normal gradient preventing separation, and thus creates lift by suction peaking on top of the leading edge. The slip boundary condition prevents separation and downwash is created with the help of the low-pressure wake of streamwise vorticity at rear separation. The high pressure at the leading edge moves further down and the pressure below increases slowly, contributing to the main lift coming from suction above. The net drag from the upper surface is close to zero because of the negative drag at the leading edge, known as *leading edge suction*, while the drag from the lower surface increases (linearly) with the angle of the incoming flow, with somewhat increased but still small drag slope. This explains why the line to a flying kite can be almost vertical even in strong wind, and that a thick wing can have less drag than a thin.

Phase 3: $12 \leq \alpha \leq 14$

Beginning stall with lift non-increasing while drag is increasing super-linearly.

Phase 4: $14 \leq \alpha \leq 16$

Stall with rapidly increasing drag.

14.1 Summary Lift and Drag

The lift generation in Phase 1 and 3 can rather easily be envisioned, while both the lift and drag in Phase 2 results from a (fortunate) intricate interplay of stability and instability of potential flow: The main lift comes from upper surface suction arising from a turbulent boundary layer with small skin friction combined with rear separation instability generating low-pressure streamwise vorticity, while the drag is kept small by negative drag from the leading edge.

We can thus summarize as follows:

- Substantial lift from suction on upper surface.
- Small drag from suction on leading edge.

Chapter 15

Computation vs Experiments

The computations shown are obtained by the finite element solver $G2$ with automatic mesh adaption from a posteriori error estimation of lift and drag by based on sensitivity information obtained by solving a linearized dual Navier-Stoke problem.

15.1 Meshes

We show below computational automatically adapted 3d meshes with up to 800.000 mesh points together with snapshots of computed turbulent velocities.

15.2 Data for Experiments

Ladson 1
$Re = 8.95e6, M = 0.15$ Grit level 60W (wraparound). grit acts to trip the boundary layer.

Ladson 2
$Re = 6.00e6, M = 0.15$ Grit level 60W (wraparound).

Ladson 3
$Re = 8.95e6, M = 0.30$ Grit level 120W (wraparound).

Gregory/O'Reilly
$Re = 2.88e6, M = 0.16$

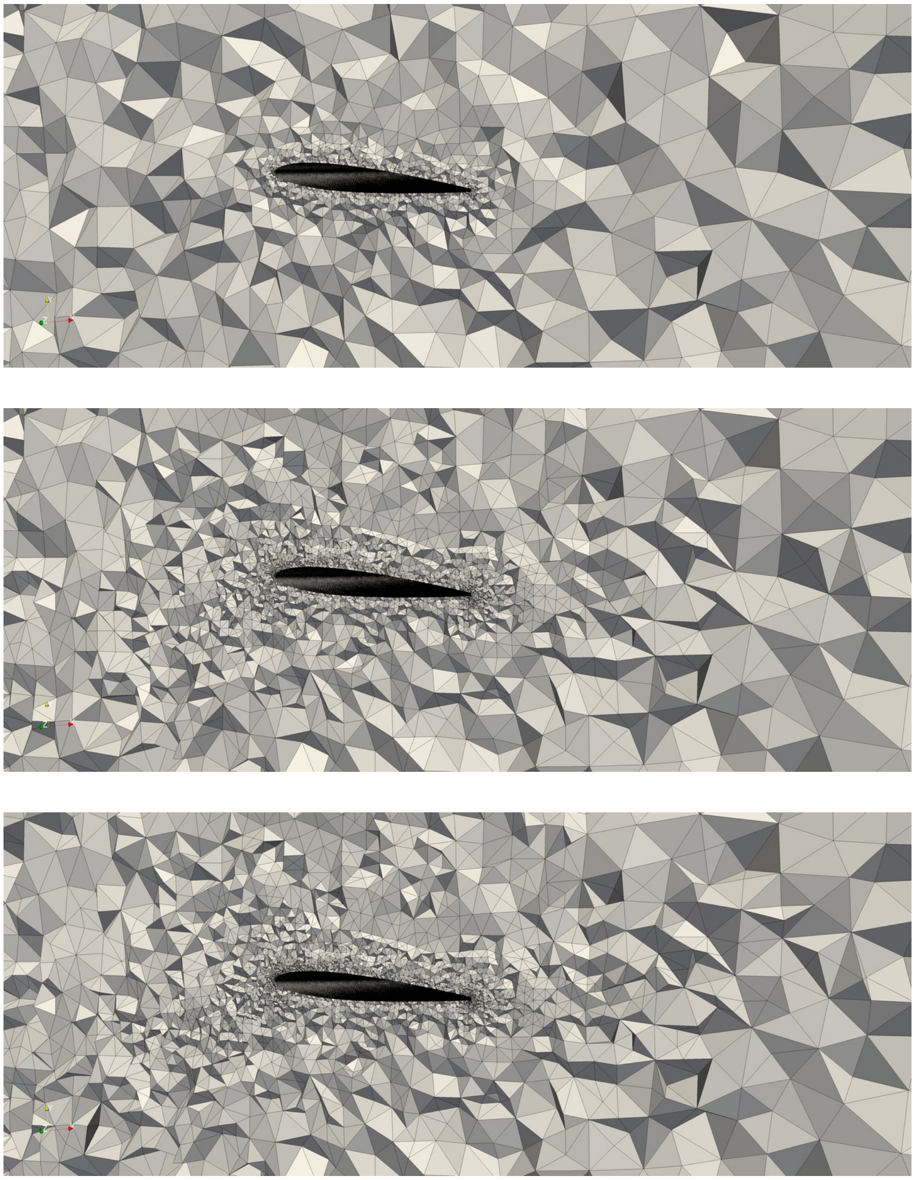


Figure 15.1: Automatically adapted meshes for $aoa = 10$, with inital mesh top, iteration 4 center, and iteration 8 bottom.

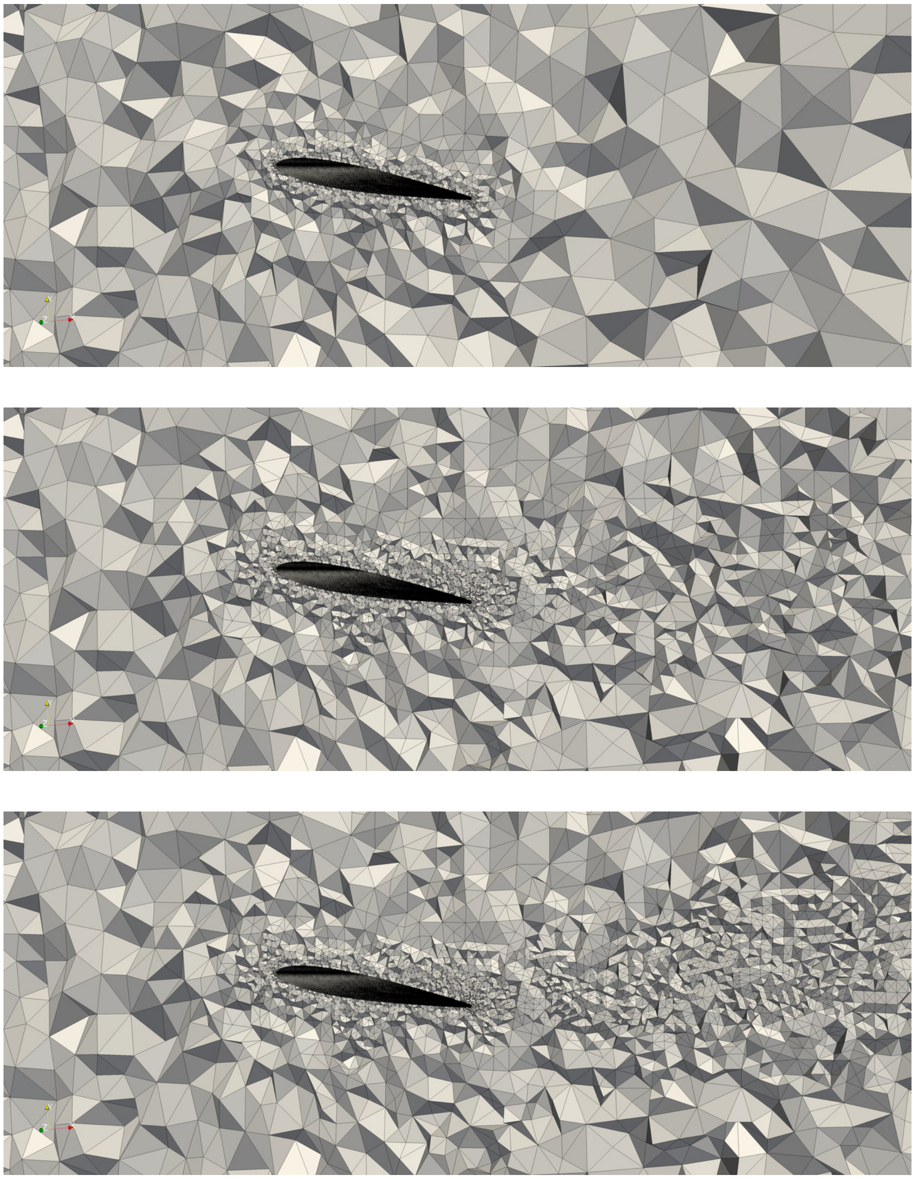


Figure 15.2: Automatically adapted meshes for $\alpha = 14$, with mesh top, iteration 4 center and iteration 8 bottom.

where the wraparound grit in the Ladson experiments acts to trip the boundary layer.

15.3 Comparing Computation with Experiment

Fig. **??** shows that G2 computations lie within variations in experiments [55, 58]. (Valid point?)

Computations more precise than experiment:

G2 with slip models a very large Reynolds number representative of e.g. a jumbojet, which cannot be attained in wind tunnel where scale model are used. Upscaling of test results is cumbersome because boundary layers do not scale. This means that computations can be closer to reality than wind tunnel experiments.

Of particular importance is the maximal lift coefficient, which cannot be predicted by Kutta-Zhukovsky nor in model experiments.

Chapter 16

Sensitivity of Lift/Drag vs Discretization

16.1 A Posteriori Error Control by Duality

G2 is equipped with automatic a posteriori error estimation in chosen output quantities such as lift and drag, which expresses the sensitivity of lift and drag with respect to the residual of a computed Navier-Stokes solution in terms of a weight function obtained by solving a dual linearized Navier-Stokes equation with the size of derivatives of the dual velocity and pressure representing the weight. G2 automatically adapts the computational mesh according to the weight so as to optimize computational resources.

16.2 Dual Pressure and Velocity

In Fig. 16.1-16.4 we show dual pressure and velocity for lift/drag indicating mesh refinement where derivatives are large.

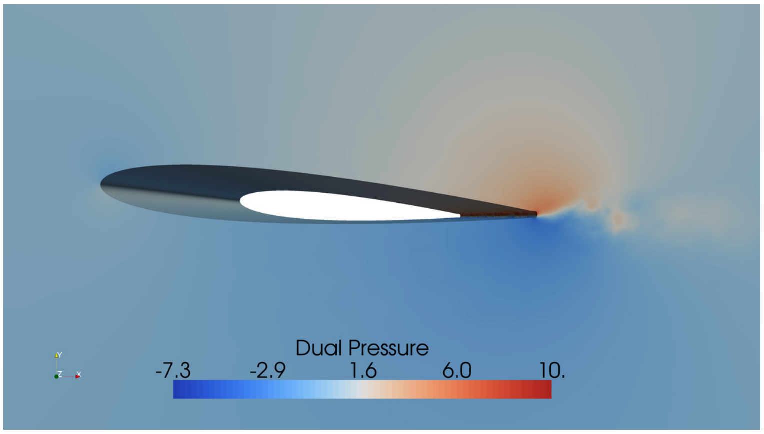


Figure 16.1: Dual pressure $\alpha = 4$.

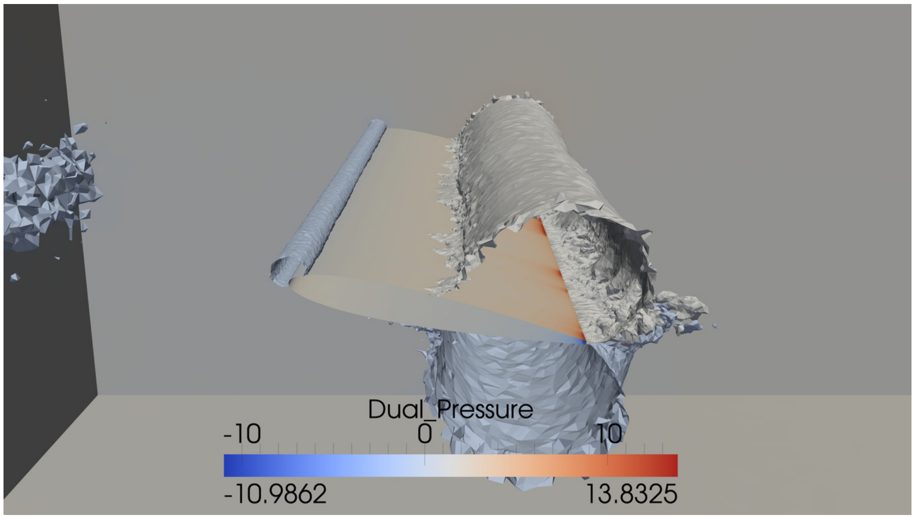


Figure 16.2: Dual pressure $\alpha = 12$.

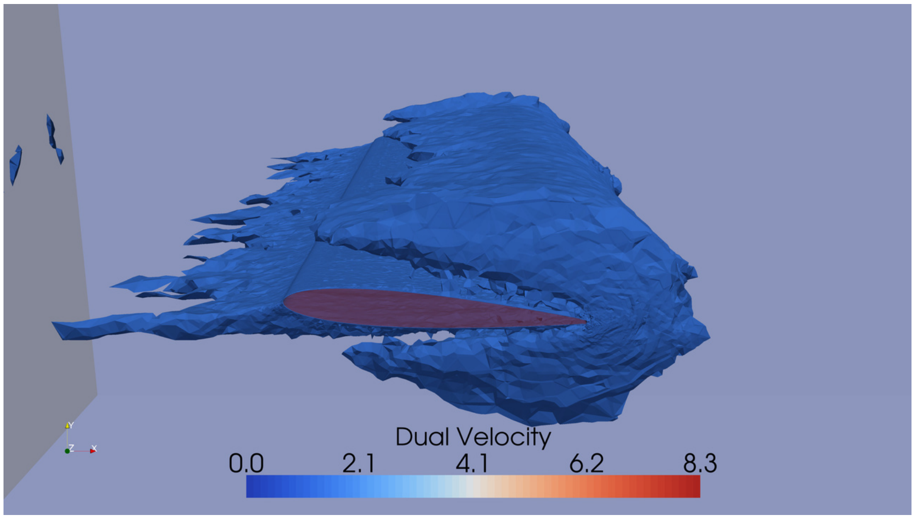


Figure 16.3: Dual velocity $\alpha = 4$.

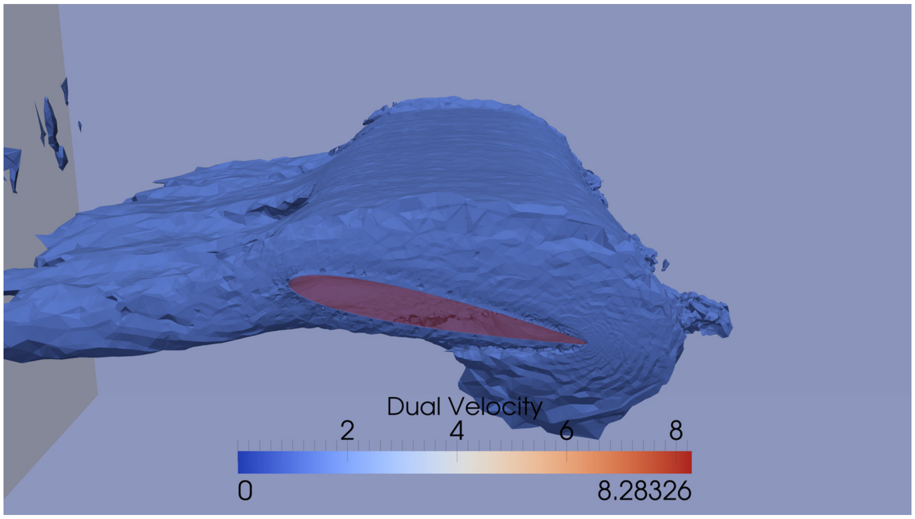


Figure 16.4: Dual velocity $\alpha = 12$.

Chapter 17

Preparing Understanding

17.1 New Resolution of d'Alembert's Paradox

We have said that the new flight theory comes out of a *new resolution* of d'Alembert's paradox [103, 104, 102]. The new resolution is based on a stability analysis showing that zero-lift/drag potential flow is *unstable* and in both computation and reality is replaced by turbulent flow with both lift and drag.

The new resolution is fundamentally different from the classical official resolution attributed to Prandtl [120, 124, 137], which disqualifies potential flow because it satisfies a *slip* boundary condition allowing fluid particles to glide along the boundary without friction force, and does not satisfy a *no-slip* boundary condition requiring the fluid particles to stick to the boundary with zero relative velocity and connect to the free-stream flow through a thin *boundary layer*, as demanded by Prandtl.

17.2 Slip/Small Friction Boundary Conditions

In contrast to Prandtl, we complement in the new theory Navier-Stokes equations with a *friction force boundary condition* for *tangential forces* on the boundary with a *small friction coefficient* as a model of the *small skin friction* resulting from a *turbulent boundary layer* of *slightly viscous flow*. In the limit of zero boundary friction this becomes a slip boundary condition, which means that potential flow can be seen as a solution of the Navier-Stokes equations subject to a small perturbation from small viscous stresses. In the new theory we then disqualify potential flow because it is unstable, that is on physical grounds, and not as Prandtl on formal grounds because it does not satisfy no-slip boundary conditions.

The Navier-Stokes equations can be complemented by no-slip or slip/friction boundary conditions, just like Poisson's equation can be complemented by Dirichlet or Neumann boundary conditions. The choice of boundary conditions depends on which data is available. In the aerodynamics of larger birds and airplanes, the skin friction is small and can be approximated by zero friction or a slip boundary condition.

For small insects viscous effects become important, which changes the physics of flight and makes gliding flight impossible. An albatross is a very good glider, while a fruit fly cannot glide at all in the syrup-like air it meets and can only move by using some form of paddling.

17.3 Computable + Correct = Secret

We solve the Navier-Stokes equations (NS) with slip/friction boundary condition using an *adaptive stabilized finite element method* with *duality-based a posteriori error control* referred to as *General Galerkin* or *G2* presented in detail in [103] and available in executable open source form from [132]. The stabilization in G2 acts as an *automatic computational turbulence model*, and the only input is the geometry of the wing. We thus find by computation that lift is not connected to circulation in contradiction to Kutta-Zhukovsky's theory and that the curse of Prandtl's laminar boundary layer theory (also questioned in [92, 93, 72]) can be circumvented. Altogether, we show in this book that *ab initio* computational fluid mechanics opens new possibilities of flight simulation ready to be explored and utilized.

- NS with no-slip: uncomputable: hides the secret,
- NS with slip: computable: reveals the secret.

17.4 No Lift without Drag

The zero-drag of potential flow has been (and still is) leading aerodynamicis to search for wings with lift but without drag [3], which we have seen is not a feature of Navier-Stokes solutions, and thus is unrealistic.

Chapter 18

Kutta-Zhukovsky-Prandtl Incorrect

The Kutta-Zhukovsky theory states that substantial lift comes from substantial circulation, but does not tell how the wing manages to cause substantial circulation of air around itself. Computing solutions of the Navier-Stokes equations shows that the wing does not do that as shown in Fig. 18.1, where the circulation is computed around the curve of the section of the section of the wing: The lift is substantial and increases linearly with the angle of attack until stall begins, while the circulation remains small. Evidently, lift does not originate from circulation.

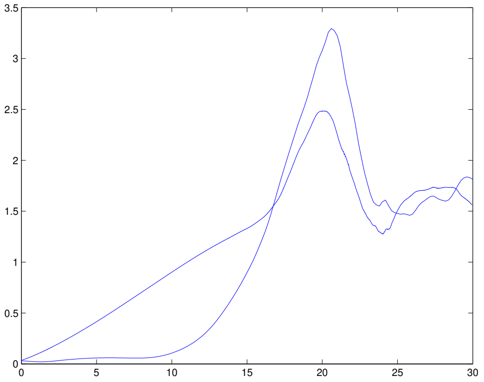


Figure 18.1: Lift coefficient and circulation as functions of the angle of attack.

Chapter 19

Mathematical Miracle of Sailing

> By denying scientific principles, one may maintain any paradox. (Galileo Galilei)

Both the sail and keel of a sailing boat under tacking against the wind, act like wings generating lift and drag, but the action, geometrical shape and angle of attack of the sail and the keel are different. The effective angle of attack of a sail is typically 15-20 degrees and that of a keel 5-10 degrees, for reasons which we now give.

The boat is pulled forward by the sail, assuming for simplicity that the beam is parallel to the direction of the boat at a minimal tacking angle, by the component $L\sin(15)$ of the lift L, as above assumed to be perpendicular to the effective wind direction, but also by the following contributions from the drag assumed to be parallel to the effective wind direction: The negative drag on the leeeward side at the leading edge close to the mast gives a positive pull which largely compensates for the positive drag from the rear leeward side, while there is less positive drag from the windward side of the sail as compared to a wing profile, because of the difference in shape. The result is a forward pull $\approx \sin(15)L \approx 0.2L$ combined with a side (heeling) force $\approx L\cos(15) \approx L$, which tilts the boat and needs to be balanced by lift from the the keel in the opposite direction. Assuming the lift/drag ratio for the keel is 13, the forward pull is then reduced to $\approx (0.2 - 1/13)L \approx 0.1L$, which can be used to overcome the drag from the hull minus the keel.

The shape of a sail is different from that of a wing which gives smaller drag from the windward side and thus improved forward pull, while the keel has the shape of a symmetrical wing and acts like a wing. A sail with aoa $15-20$ degrees gives maximal pull forward at maximal heeling/lift with contribution also from the rear part of the sail, like for a wing just before stall, while the drag is smaller than for a wing at 15-20 degrees aoa (for which the lift/drag ratio is 4-3), with the motivation given above. The lift/drag curve for a sail is thus different from that of wing with lift/drag ratio at aoa 15-20 much larger for a sail. On the other hand, a keel with aoa 5-10 degrees has a lift/drag ratio about 13. A sail at aoa 15-20 thus gives maximal pull at strong heeling force and small drag, which together with a keel at aoa 5-10 with strong lift and small drag, makes an efficient combination. This explains why modern designs combine a deep narrow keel acting efficiently for small aoa, with a broader sail acting efficiently at a larger aoa.

Using a symmetrical wing as a sail would be inefficient, since the lift/drag ratio is poor at maximal lift at aoa 15-20. On the other hand, using a sail as a wing can only be efficient at a large angle of attack, and thus is not suitable for cruising. This material is developed in more detail in [108].

Part III

Solving Navier-Stokes

Chapter 20

Navier-Stokes Equations

> Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations. (Clay Mathematics Institute Millennium Problem [106])

20.1 Conservation of Mass, Momentum and Energy

The basic mathematical model of fluid mechanics takes the form of the *Navier-Stokes equations* expressing conservation of mass, momentum and energy of a viscous fluid in the conservation variables of density, momentum and energy with the viscosity as a given coefficient. For an incompressible fluid the equations can be formulated in terms velocity and pressure, referred to as the *incompressible Navier-Stokes equations*, with a decoupled energy equation.

The fluid mechanics of subsonic flight is modeled by the incompressible Navier-Stokes equations for a slightly viscous fluid, which is the focus of this book. The *Reynolds number* $Re = \frac{UL}{\nu}$ where U is a characteristic fluid velocity, L a characteristic length and ν the fluid viscosity, is used to identify the high Reynolds number flow occuring in aerodynamics with Re of size 10^8 for large airplanes.

The incompressible Navier-Stokes equations are complemented by initial values for velocity, and boundary conditions specifying either velocities or forces on the boundary. A *no-slip* boundary condition sets the fluid velocity to zero on the boundary, while a *slip* boundary condition sets the velocity normal to the boundary to zero together with the tangential (friction) force. The slip condition is a limit case of a combined normal velocity-tangential stress boundary condition with the tangential stress set to zero as a model of zero skin friction.

The Navier-Stokes equations for an incompressible fluid of unit density with *small viscosity* $\nu > 0$ and *small skin friction* $\beta \geq 0$ filling a volume Ω in $\mathbb{R}^3$ surrounding a solid body with boundary Γ over a time interval $I = [0, T]$, read as follows: Find the velocity $u = (u_1, u_2, u_3)$ and pressure p depending on $(x, t) \in \Omega \cup \Gamma \times I$, such that

$$\begin{aligned}
\dot{u} + (u \cdot \nabla)u + \nabla p - \nabla \cdot \sigma &= f && \text{in } \Omega \times I, \\
\nabla \cdot u &= 0 && \text{in } \Omega \times I, \\
u_n &= g && \text{on } \Gamma \times I, \\
\sigma_s &= \beta u_s && \text{on } \Gamma \times I, \\
u(\cdot, 0) &= u^0 && \text{in } \Omega,
\end{aligned} \tag{20.1}$$

where u_n is the fluid velocity normal to Γ, u_s is the tangential velocity, $\sigma = 2\nu\epsilon(u)$ is the viscous (shear) stress with $\epsilon(u)$ the usual velocity strain, σ_s is the tangential stress, f is a given volume force, g is a given inflow/outflow velocity with $g = 0$ on a non-penetrable boundary, and u^0 is a given initial condition.

We notice the skin friction boundary condition coupling the tangential stress σ_s to the tangential velocity u_s with the friction coefficient β with $\beta = 0$ for slip, and $\beta >> 1$ for no-slip. We note that β is related to the standard *skin friction coefficient* $c_f = \frac{2\tau}{U^2}$ with τ the tangential stress per unit area, by the relation $\beta = \frac{U}{2} c_f$. In particular, β tends to zero with c_f (if U stays bounded).

Prandtl insisted on using a no-slip velocity boundary condition with $u_s = 0$ on Γ, because his resolution of d'Alembert's paradox hinged on discriminating potential flow by this condition. On the oher hand, with the new resolution of d'Alembert's paradox, relying instead on instability of potential flow, we are free to choose instead a friction force boundary condition, if data is available. Now, experiments show [124, 28] that the skin friction coefficient decreases with increasing Reynolds number Re as $c_f \sim Re^{-0.2}$, so that $c_f \approx 0.0005$ for $Re = 10^{10}$ and $c_f \approx 0.007$ for $Re = 10^5$. Accordingly we model a turbulent boundary layer by a friction boundary condition with a friction parameter $\beta \approx 0.03 U Re^{-0.2}$. For very large Reynolds numbers, we can effectively use $\beta = 0$ in G2 computation corresponding to slip boundary conditions.

20.2 Wellposedness and Clay Millennium Problem

The mathematician J. Hadamard identified in 1902 [95] *wellposedness* as a necessary requirement of a solution of a mathematical model, such as the Navier-Stokes equations, in order to have physical relevance: Only *wellposed* solutions which are suitably stable in the sense that small perturbations have small effects when properly measured, have physical significance as observable pheonomena.

Leray's requirement of wellposedness is absolutely fundamental, but the question whether solutions of the Navier-Stokes equations are wellposed, has not been studied because of lack mathematical techniques for quantitative analysis. This is evidenced in the formulation of the Clay Millennium Prize Problem on the Navier-Stokes equations excluding wellposedness [106, 102].

The mathematical Garret Birkhoff became heavily criticized for posing this question in [92], which stopped him from further studies. The first step towards resolution of d'Alembert's paradox and the mathematical secret of flight is thus to pose the question if potential flow is wellposed, and then to realize that it is not. It took 256 years to take these steps.

20.3 Laminar vs Turbulent Boundary Layer

As developed in more detail in [134], we make a distinction between laminar (boundary layer) separation modeled by no-slip and turbulent (boundary layer) separation modeled by slip/small friction. Note that laminar separation cannot be modeled by slip, since a laminar boundary layer needs to be resolved with no-slip to get correct (early) separation. On the other hand, as will be seen below, in turbulent (but not in laminar) flow the interior turbulence dominates the skin friction turbulence indicating that the effect of a turbulent boundary layer can be modeled by slip/small friction, which can be justified by an posteriori sensitivity analysis as shown in [134].

We thus assume that the boundary layer is turbulent and is modeled by slip/small friction, which effectively includes the case of laminar separation followed by reattachment into a turbulent boundary layer.

Chapter 21

G2 Computational Solution

21.1 General Galerkin G2: Finite Element Method

We show in [103, 102, 104] that the Navier-Stokes equations (20.1) can be solved by a weighted least squares residual stabilized finite element referred to as *General Galerkin* or *G2*.

Writing the Navier-Stokes equations in symbolic form as $R(u, p) = 0$, the G2 method determines a piecewise linear computational solution (U, P) on a given finite element mesh such that the residual $R(U, P)$ is small in a mean-value sense (Galerkin property) and with a certain weighted control of $R(U, P)$ in a least-square sense (residual stabilization).

G2 produces turbulent solutions characterized by substantial turbulent dissipation from the least squares residual stabilization acting as an automatic turbulence model, reflecting that $R(U, P)$ cannot be made small pointwise in turbulent regions.

G2 is equipped with automatic a posteriori error control guaranteeing correct lift and drag coefficients [103, 34, 33, 38, 99, 100] up to an error tolerance of a few percent on meshes with a few hundred thousand or million mesh points number of mesh points depending on geometry complexity.

G2 with slip is thus capable of modeling slightly viscous turbulent flow with $Re > 10^6$ of relevance in many applications in aero/hydro dynamics, including flying, sailing, boating and car racing, with possible millions of mesh points, to be compared with the impossible quadrillions required [43] for boundary layer resolution.

G2 with slip works because slip is good model of the turbulent boundary layer od slightly viscous flow, and interior turbulence does not have to be resolved to physical scales to capture mean-value outputs [103].

G2 with slip thus offers a wealth of infomation at affordable cost, while Prandtl's requirement of boundary layer resolution cannot be met by any forseeable computer.

21.2 A Posteriori Error Control and Wellposedness

G2 is an adaptive finite element method with duality-residual based error control of the principal form

$$dM(U,P) \leq S\|hR(U,P)\| \tag{21.1}$$

$dM(U,P)$ is the variation of a certain mean-value output $M(U,P)$ such as lift or drag coeffiients of a computed Navier-Stokes solution (U,P) with Navier-Stokes residual $R(U,P)$ on a mesh with mesh size h and S is a stability factor measuring certain norms of an associated dual solution and $\|\cdot\|$ is a mean square integral norm.

If in a G2 computation $S\|hR(U,P) < TOL$, the output $M(U,P)$ is guaranteed to change less than TOL under any mesh refinement.

21.3 What You Need to Know

To understand flight it is not necessary to get into the details of G2; it is sufficient to understand that G2 solves the Navier-Stokes equations with an automatic control of the the computational error, which guarantees that G2 solutions are proper solutions capable of unraveling the secrets hidden in the Navier-Stokes equations, thereby offering valuable information for both understanding and design.

21.4 Turbulent Flow around a Car

In Fig. 21.1 we show computed turbulent G2 flow around a car with substantial drag in accordance with wind-tunnel experiments. We see a pattern of streamwise vorticity forming in the rear wake. We also see surface vorticity forming on the hood transversal to the main flow direction. We will discover similar features in the flow of air around a wing...

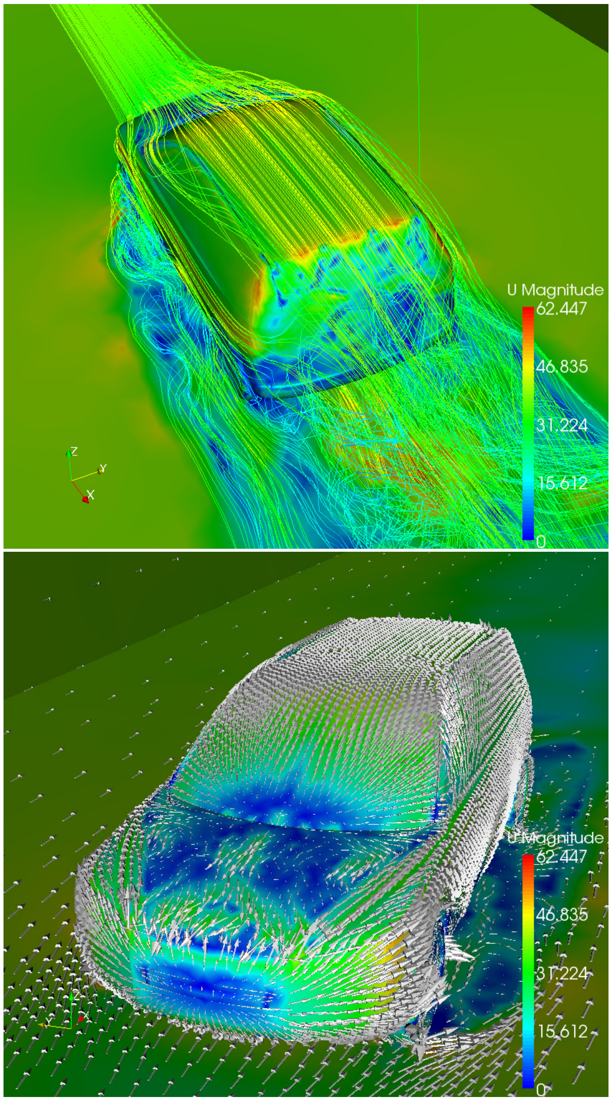


Figure 21.1: Velocity of turbulent flow around a car

Chapter 22

Potential Flow

> Because of d'Alembert's paradox) fluid mecahnics was from start split into the field of hydraulics, observing phenomena which could not be explained, and mathematical or theoretical fluid mechanics explaining phenomena which could not be observed. (Chemistry Nobel Laureate Sir Cyril Hinshelwood [63])

22.1 Euler defeated by d'Alembert

Mathematical fluid mechanics started when Euler in the 1740s discovered certain solutions in two space dimensions of the Navier-Stokes equations with vanishingly small viscosity, with velocities of the form $u = \nabla\varphi$, where the potential φ is a harmonic function satisfying $\Delta\varphi = 0$ in the fluid. These were named *potential solutions* characterized as

- inviscid: vanishingly small viscosity,
- incompressible: $\nabla \cdot u = 0$,
- irrotational: $\nabla \times u = 0$
- stationary: $\dot{u} = 0$.

This promised a fluid mechanics boom for mathematicians as experts of harmonic functions, but the success story quickly collapsed when d'Alembert in 1752 showed that both lift and drag of potential solutions are zero, which showed that the wonderful potential solutions were unphysical and thus were doomed as useless. But potential solution were essentially the only solutions which could be constructed analytically, which led to a long-lasting split of fluid mechanics into:

- practical fluid mechanics or hydraulics: observing phenomena which cannot be explained (non-zero lift and drag),
- theoretical fluid mechanics: explaining phenomena which cannot be observed (zero drag and lift),

according to the above quote.

We shall see that it took 254 years to resolve the paradox, but once it was resolved the mystery of flight could be uncovered without split between theory and reality.

22.2 Potential Flow as Near Navier-Stokes Solution

Potential flow (u, p) with velocity $u = \nabla\varphi$, where φ is harmonic in Ω and satisfies a homogeneous Neumann condition on Γ and suitable conditions at infinity, can be seen as a solution of the Navier-Stokes equations for slightly viscous flow with slip boundary condition, subject to

- perturbation of the volume force $f = 0$ in the form of $\sigma = \nabla \cdot (2\nu\epsilon(u))$,
- perturbation of zero friction in the form of $\sigma_s = 2\nu\epsilon(u)_s$,

with both perturbations being small because ν is small and a potential flow velocity u is smooth. Potential flow can thus be seen as a solution of the Navier-Stokes equations with small force perturbations tending to zero with the viscosity. We can thus express d'Alembert's paradox as the zero lift/drag of a Navier-Stokes solution in the form of a potential solution, and resolve the paradox by realizing that potential flow is unstable and thus cannot be observed as a physical flow.

Potential flow is like an inverted pendulum, which cannot be observed in reality because it is unstable and under infinitesimal perturbations turns into a swinging motion. A stationary inverted pendulum is a fictious mathematical solution without physical correspondence because it is unstable. You can only observe phenomena which in some sense are stable, and an inverted pendelum or potential flow is not stable in any sense.

22.3 Potential Flow Separates only at Stagnation

Potential flow has the following crucial property which partly will be inherited by real turbulent flow, and which explains why a flow over a wing subject to small skin friction can avoid separating at the crest and thus generate downwash, unlike viscous flow with no-slip, which separates at the crest without downwash. We will conclude that gliding flight is possible only in slightly viscous incompressible flow. For simplicity we consider two-dimensional potential flow around a cylindrical body such as a long wing (or cylinder).

Theorem. Let φ be harmonic in the domain Ω in the plane and satisfy a homogeneous Neumann condition on the smooth boundary Γ of Ω. Then the streamlines of the corresponding velocity $u = \nabla\varphi$ can only separate from Γ at a point of stagnation with $u = \nabla\varphi = 0$.
Proof. Let ψ be a harmonic conjugate to φ with the pair (φ, ψ) satisfying the Cauchy-Riemann equations (locally) in Ω. Then the level lines of ψ are the streamlines of φ and vice versa. This means that as long as $\nabla\varphi \neq 0$, the boundary curve Γ will be a streamline of u and thus fluid particles cannot separate from Γ in bounded time.

22.4 Vortex Stretching

Formally applying the curl operator $\nabla\times$ to the momentum equation of (20.1), with $\nu = \beta = 0$ for simplicity, we obtain the *vorticity equation*

$$\dot{\omega} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \nabla\times f \quad \text{in } \Omega, \tag{22.1}$$

which is a convection-reaction equation in the vorticity $\omega = \nabla\times u$ with coefficients depending on u, of the same form as the linearized equation (**??**), with similar properties of exponential perturbation growth $\exp(|\nabla u|t)$ referred to as *vortex stretching*. Kelvin's theorem formally follows from this equation assuming the initial vorticity is zero and $\nabla\times f = 0$ (and $g = 0$), but exponential perturbation growth makes this conclusion physically incorrect: We will see below that large vorticity can develop from irrotational potential flow even with slip boundary conditions. , by properly understanding both the physical and unphysical aspects of potential solutions.

Chapter 23

D'Alembert and his Paradox

> To those who ask what the infinitely small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be. (Leonhard Euler)
>
> High office, is like a pyramid; only two kinds of animals reach the summit–reptiles and eagles. (d'Alembert)
>
> Just go on . . . and faith will soon return. (d'Alembert to a friend hesitant with respect to infinitesimals)
>
> If one looks at all closely at the middle of our own century, the events that occupy us, our customs, our achievements and even our topics of conversation, it is difficult not to see that a very remarkable change in several respects has come into our ideas; a change which, by its rapidity, seems to us to foreshadow another still greater. Time alone will tell the aim, the nature and limits of this revolution, whose inconveniences and advantages our posterity will recognize better than we can. (d'Alembert on the Enlightment)

23.1 d'Alembert and Euler and Potential Flow

Working on a 1749 Prize Problem of the Berlin Academy on flow drag, d'Alembert was led to the following contradiction referred to as *d'Alembert's paradox* [75, 76, 77, 78, 83] between observation and theoretical prediction:

- *It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers to elucidate.*

The great mathmatician *Leonard Euler*(1707-1783) had come to same conclusion of zero drag of potential flow in his work on gunnery [79] from 1745 based on the observation that in potential flow the high pressure forming in front of the body is balanced by an equally high pressure in the back, in the case of a boat moving through water expressed as

- *...the boat would be slowed down at the prow as much as it would be pushed at the poop...*

This is the idea of Aristotle adopted by da Vinci, which we met above in the form of peristaltic motion.

More precisely, d'Alembert's paradox concerns the contradiction between observations of substantial drag/lift of a body moving through a slightly viscous fluid such as air and water, with the mathematical prediction of zero drag/lift of potential flow defined as inviscid, incompressible, irrotational and stationary flow. Evidently, flying is incompatible with potential flow, and in order to explain flight d'Alembert's paradox had to be resolved. But d'Alembert couldn't do it and all the great mathematical brains of the 18th and 19th century stumbled on it: Nobody could see that any of the assumptions (i)-(iv) were wrong and the paradox remained unsolved. We shall resolve the paradox below and find the true reason that potential flow with zero drag/lift is never observed. And the true reason is not (iii).

We recall that a flow is *irrotational* if the flow velocity u has zero *vorticity*, that is if $\nabla \times u = 0$, in which case (for a simply connected domain) the velocity u is given as the gradient of a potential function: $u = \nabla\varphi$ where φ is the potential. If u is also incompressible, then

$$\Delta\varphi = \nabla \cdot \nabla\varphi = \nabla \cdot u = 0$$

and thus the potential φ is a *harmonic function* satisfying *Laplace's equation*:

$$\Delta\varphi = 0. \tag{23.1}$$

This promised to open fluid mechanics for take-over by harmonic functions in the hands of mathematicians, supported by *Kelvin's theorem* stating that without external forcing, an incompressible inviscid flow will stay irrotational if initiated as irrotational. Mathematicians thus expected to find an abundance of potential flows governed by harmonic potentials in the fluid mechanics of slightly viscous flow, but such flows did not seem to appear in reality, and nobody could understand why.

Figure 23.1: D'Alembert formulating his paradox.

23.2 The Euler Equations

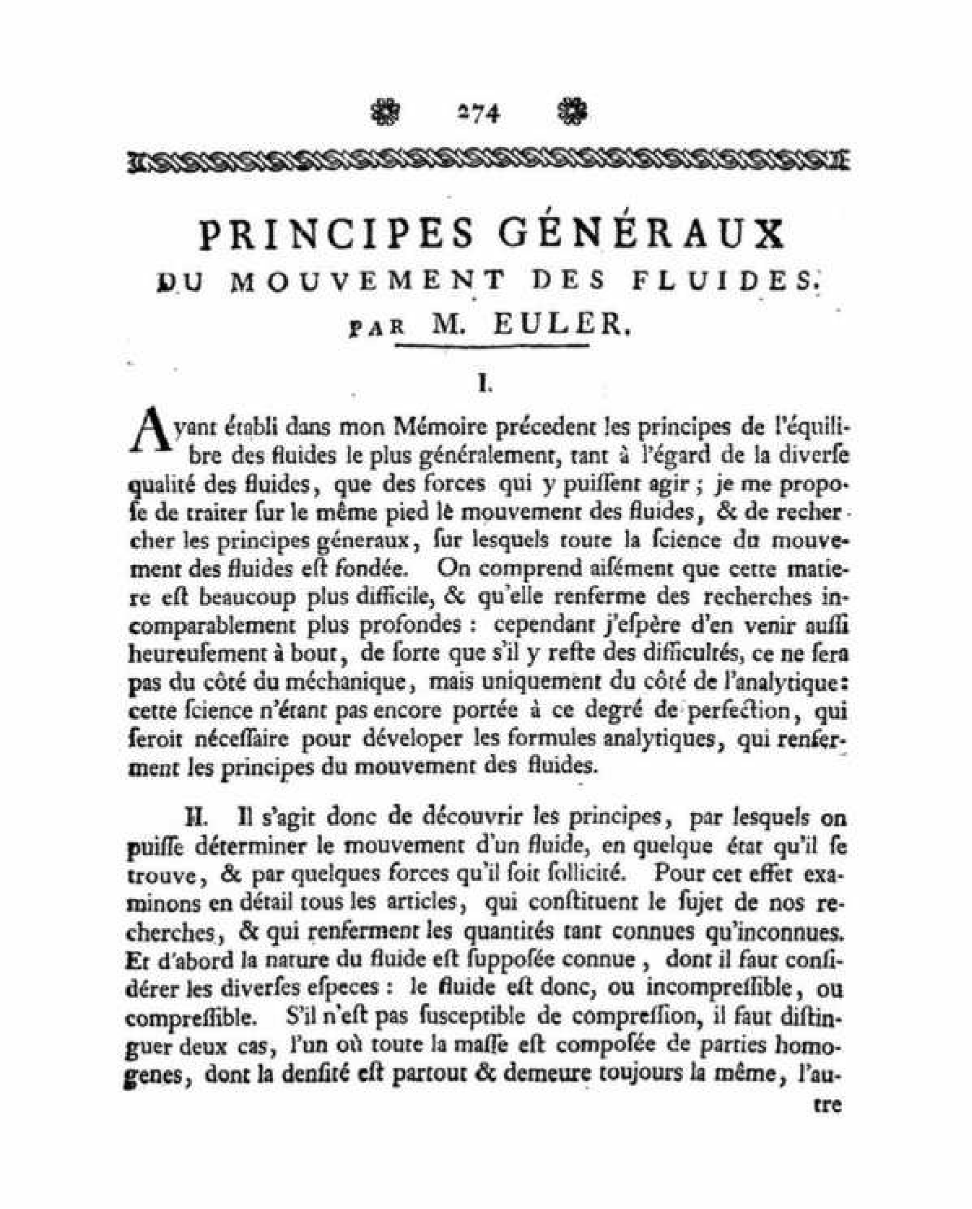
The basic equations in fluid mechanics expressing *conservation of momentum* or *Newton's 2nd law* connecting force to accelleration combined with *conservation of mass* in the form of incompressibility, were formulated by Euler in 1755 as the *Euler equations* for an *incompressible inviscid* fluid (of unit density) enclosed in a volume Ω in $\mathbb{R}^3$ with boundary Γ: Find the velocity $u = (u_1, u_2, u_3)$ and pressure p such that

$$\begin{aligned} \dot{u} + (u \cdot \nabla)u + \nabla p &= f && \text{in } \Omega \times I, \\ \nabla \cdot u &= 0 && \text{in } \Omega \times I, \\ u \cdot n &= g && \text{on } \Gamma \times I, \\ u(\cdot, 0) &= u^0 && \text{in } \Omega, \end{aligned} \tag{23.2}$$

where the dot signifies differentiation with respect to time, n denotes the outward unit normal to Γ, f is a given volume force, g is a given inflow/outflow velocity, u^0 is a given initial condition and $I = [0, T]$ a given time interval. We notice the *slip boundary condition* $u \cdot n = 0$ modeling a non-penetrable boundary with zero friction.

The momentum equation can alternatively be formulated as

$$\dot{u} + \nabla(\frac{1}{2}|u|^2 + p) + u \times \omega = f \tag{23.3}$$



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PRINCIPES GÉNÉRAUX
DU MOUVEMENT DES FLUIDES.
PAR M. EULER.

I.

Ayant établi dans mon Mémoire précedent les principes de l'équilibre des fluides le plus généralement, tant à l'égard de la diverſe qualité des fluides, que des forces qui y puiſſent agir ; je me propoſe de traiter ſur le même pied le mouvement des fluides, & de rechercher les principes géneraux, ſur lesquels toute la ſcience du mouvement des fluides eſt fondée. On comprend aiſément que cette matiere eſt beaucoup plus difficile, & qu'elle renferme des recherches incomparablement plus profondes : cependant j'eſpère d'en venir auſſi heureuſement à bout, de ſorte que s'il y reſte des difficultés, ce ne ſera pas du côté du méchanique, mais uniquement du côté de l'analytique: cette ſcience n'étant pas encore portée à ce degré de perfection, qui ſeroit néceſſaire pour déveloper les formules analytiques, qui renferment les principes du mouvement des fluides.

II. Il s'agit donc de découvrir les principes, par lesquels on puiſſe déterminer le mouvement d'un fluide, en quelque état qu'il ſe trouve, & par quelques forces qu'il ſoit ſollicité. Pour cet effet examinons en détail tous les articles, qui conſtituent le ſujet de nos recherches, & qui renferment les quantités tant connues qu'inconnues. Et d'abord la nature du fluide eſt ſuppoſée connue, dont il faut conſidérer les diverſes eſpeces : le fluide eſt donc, ou incompreſſible, ou compreſſible. S'il n'eſt pas ſusceptible de compreſſion, il faut diſtinguer deux cas, l'un où toute la maſſe eſt compoſée de parties homogenes, dont la denſité eſt partout & demeure toujours la même, l'autre

Figure 23.2: First page of Euler's *General Principles concerning the Motion of Fluids* from 1757 [80].

where

$$\omega = \nabla \times u$$

is the *vorticity* of the velocity u, which follows from the following calculus identity:

$$\frac{1}{2}\nabla|u|^2 = (u \cdot \nabla)u + u \times (\nabla \times u).$$

For a stationary irrotational velocity u with $\dot{u} = 0$ and $\omega = \nabla \times u = 0$, we find that if $f = 0$, then

$$\frac{1}{2}|u|^2 + p = C \tag{23.4}$$

where C is a constant, which is nothing but *Bernouilli's principle* coupling small velocity to large pressure and vice versa.

We conclude that a potential flow velocity $u = \nabla\varphi$ solves the Euler equations with the pressure p given by Bernouilli's law.

Kelvin's theorem states that if the initial velocity u^0 is irrotational and $\nabla \times f = 0$ and $g = 0$, then a smooth Euler solution velocity will remain irrotational for positive time. Below we will question the validity of Kelvin's theorem on the ground that solutions of the Euler equations in general are not smooth, even if data are.

23.3 Potential Flow around a Circular Cylinder

To understand how a wing generates lift and drag it is instructive to first consider the corresponding problem for a circular cylinder, which we can view as a wing with circular cross-section. Of course you cannot fly with such a wing, but a wing is similar to a half cylinder which can be analyzed starting with a full cylinder.

We start considering potential flow around a circular cylinder of unit radius with axis along the x_3-axis in three-dimensional space with coordinates (x_1, x_2, x_3), assuming the flow velocity is $(1, 0, 0)$ at infinity, see Fig. 26.1 showing a section of through the cylinder with the flow horisontal from left to right. We can equally well think of the cylinder moving transversally through a fluid at rest. Potential flow around the cylinder is constant in the x_3-direction and is symmetric in x_1 and x_2 with zero drag/lift with the flow velocity given as the gradient of the potential

$$\varphi(r, \theta) = (r + \frac{1}{r})\cos(\theta),$$

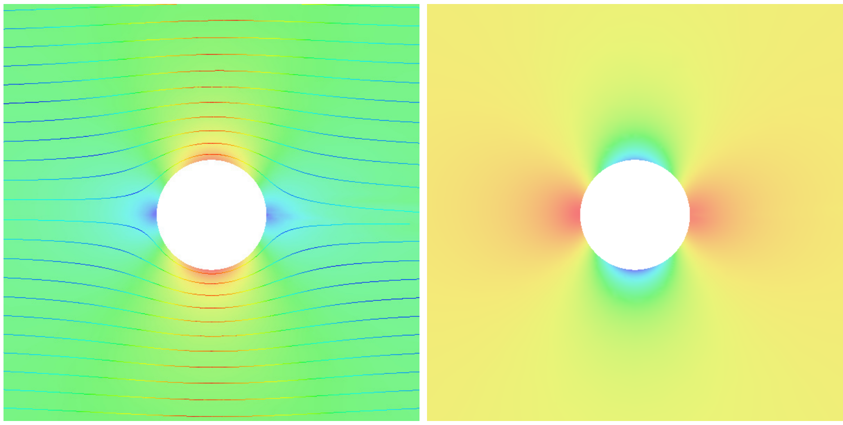


Figure 23.3: Potential flow past a circular cylinder: streamlines and fluid speed (left) and pressure (right) in a (x_1, x_2)-plane with horisontal x_1-axis in the flow direction.

in polar coordinates (r, θ) in the (x_1, x_2)-plane. The corresponding pressure (vanishing at infinity) is determined by Bernouilli's law as:

$$p = -\frac{1}{2r^4} + \frac{1}{r^2}\cos(2\theta).$$

In its simplicity potential flow is truely remarkable: It is a solution of the Euler equations for inviscid flow with slip boundary condition, which separates at the back of cylinder at the line $(1, 0, x_3)$, with equally high pressure in the front and the back and low pressure on top and bottom (with the low pressure three times as big as the high pressure), resulting in zero drag/lift. This is d'Alembert's paradox: All experience indicates that a circular cylinder subject to air flow has substantial drag, but potential flow has zero drag.

We understand that the high pressure in the back, balancing the high pressure up front, can be seen as pushing the body through the fluid according to the principle of of motion of Aristotle and da Vinci. We shall discover that there is something which is correct in this view. But the net push from behind in real flow must be smaller than in potential flow, and so real flow must be different form potential flow in the rear, but how and why? We shall see that the correct answer to these questions hide the secret of flight.

23.4 Non-Separation of Potential Flow

Direct computation shows that on the cylinder boundary

$$\frac{\partial p}{\partial n} = \frac{U^2}{R}, \tag{23.5}$$

where n is the outward unit normal to the boundary, $\frac{\partial p}{\partial n}$ is the gradient of the pressure in the unit normal direction or *normal pressure gradient* into the fluid, U is the flow speed and $R = 1$ the radius of curvature of the boundary (positive for a concave fluid domain thus positive for the cylinder). The relation (23.5) is Newton's law expressing that fluid particles gliding along the boundary must be accellerated in the normal direction by the normal pressure gradient force in order to follow the curvature of the boundary. More generally, (23.5) is the criterion for *non-separation*: Fluid particles will stay close to the boundary as long as (23.5) is satisfied, while if

$$\frac{\partial p}{\partial n} < \frac{U^2}{R}, \tag{23.6}$$

then fluid particles will separate away from the boundary tangentially. In particular, as we will see below, laminar flow separates on the crest or top/bottom of the cylinder, since the normal pressure gradient is small in a laminar boundary layer with no-slip boundary condition [124, 134].

We sum up so far: The Euler equations express conservation of mass and momentum for an inviscid incompressible fluid. Potential flow is smooth and satisfies the Euler equations. D'Alembert's paradox compares inviscid potential flow having zero drag/lift with slightly viscous flow having substantial drag/lift. Potential flow has a positive normal pressure gradient preventing separation allowing the pressure to build up on the back to push the cylinder through the fluid without drag. We shall see that this is a bit too optimistic, but only a bit; there is some push also in real (slightly viscous) flow...

Part IV

Understanding Navier-Stokes

Chapter 24

Lift and Drag from Separation

> The fear of making permanent commitments can change the mutual love of husband and wife into two loves of self - two loves existing side by side, until they end in separation. (Pope John Paul II)

We now turn to a detailed mathematical analysis of flow separation, which we will find uncovers the secret of generation of both lift and drag of a body moving through air such as a wing. We know that the flow around the body *attaches* somewhere in the front, typically around a *point of stagnation*, where the flow velocity is zero, and *separates* somewhere somehow in the rear. In many cases attachment is governed by smooth (laminar) potential flow, while separation effectively is a generator of turbulence. We shall thus find that drag can be seen as a "cost of separation", which for a wing also pays for generating lift.

We will present a scenario for separation in slightly viscous turbulent flow, which is fundamentally different from the scenario for viscous laminar flow by Prandtl based on adverse pressure gradients retarding the flow to stagnation at separation. We make a distinction betweeen separation from a laminar boundary layer with no-slip boundary condition and from a turbulent boundary layer with slip. We thus make a distinction between

- **laminar separation with no-slip in (very) viscous flow**

considered by Prandtl of relevance for viscous flow, and

- **turbulent separation with slip in slightly viscous flow**

of relevance in aerodynamics.

We noted above that separation occurs if $\frac{\partial p}{\partial n} < \frac{U^2}{R}$, where $\frac{\partial p}{\partial n}$ is the pressure gradient normal to the boundary into the fluid, U is a flow speed close to the boundary and R the curvature of the boundary, positive for a convex body. We note that in a laminar boundary layer $\frac{\partial p}{\partial n} > 0$ only in contracting flow, which causes separation as soon as the flow expands after the crest of the body. We observe that in a turbulent boundary layer with slip, $\frac{\partial p}{\partial n} > 0$ is possible also in expanding flow which can delay separation. We present a basic mechanism for tangential separation with slip based on instability at rear points of stagnation generating low-pressure rolls of streamwise vorticity reducing $\frac{\partial p}{\partial n}$.

We give evidence that Prandtl's boundary layer theory for laminar separation has fallen into this trap, with the unfortunate result is that much research and effort has gone into preventing *laminar separation* in flows which effectively are turbulent with *turbulent separation*. We present a scenario for turbulent separation without stagnation supported by analysis, computation and experiments, which is radically different from Prandtl's scenario for laminar separation at stagnation. The fundamental question concerns the fluid dynamics of *separation without stagnation*, since in slightly viscous flow the friction is too small to bring fluid particles to rest. We shall find an answer which connects to the familiar experience of the rotating flow through a bathtub drain, which in reality replaces the theoretically possible but unstable fully radial flow.

We will see that laminar separation with no-slip occurs at the crest of a wing without generating lift, while turbulent separation is delayed and thereby generates lift.

Chapter 25

Separation

25.1 From Unstable to Quasi-Stable Separation

We are concerned with the fundamental problem of fluid mechanics of the motion of a solid body, such as a subsonic airplane, car or boat, through a slightly viscous incompressible fluid such as air at subsonic speeds or water. We focus on incompressible flow at large Reynolds number (of size 10^6 or larger) around both bluff and streamlined bodies, which is always partly turbulent.

The basic problem is to determine the forces acting on the surface of the body from the motion through the fluid, with the *drag* being the total force in the direction of the flow and the *lift* the total force in a transversal direction to the flow.

As a body moves through a fluid initially at rest, like a car or airplane moving through still air, or equivalently as a fluid flows around a body at rest, approaching fluid particles are deviated by the body in contracting flow, switch to expanding flow at a crest and eventually leave the body. The flow is said to *attach* in the front and *separate* in the back as fluid particles approach and leave a proximity of the body surface.

In high Reynolds number slightly viscous flow the tangential forces on the surface, or *skin friction* forces are small and both drag and lift mainly result from pressure forces and the pressure distribution at turbulent separation is of particular concern.

Separation requires *stagnation* of the flow to zero velocity somewhere in the back of the body as opposing flows are meeting. Stagnation requires *retardation* of the flow, which requires a streamwise increasing pressure, or *adverse pressure gradient*. We show by a linearized stability analysis that retardation from opposing

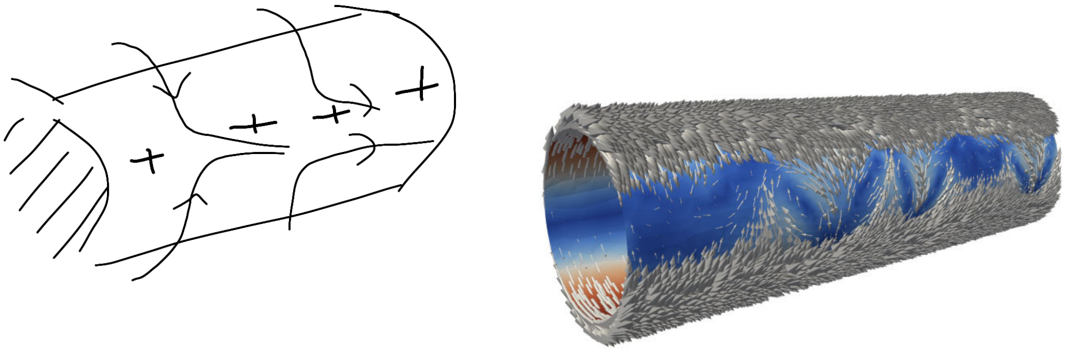


Figure 25.1: Unstable irrotational separation of potential flow around a circular cylinder (left) from line of stagnation surrounded by a high pressure zone indicated by $+$, with corresponding opposing flow instability (right).

flows is exponentially unstable, which in particular shows *potential flow* to be unstable as indictaed in Fig. 25.1. Since unstable flow cannot persist over time, we expect to find a *quasi-stable* separation pattern resulting from the most unstable mode of potential flow, as a flow without streamwise retardation from opposing flows. By quasi-stable we mean a flow which is not exponentially unstable and thus may have a certain permanence over time.

Both experiment and computation show that there is such a quasi-stable separation pattern arising from transversal reorganization of opposing potential flow in the back into a set of counter-rotating vortex tubes of swirling flow (streamwise vorticity) attaching to the body, accompanied by a zig-zag pattern of alternating low and high pressure zones around points of stagnation with low pressure inside the vortex tubes. This pattern is illustrated in Fig. 2 for a cylinder along with computation and experiment, where we see how the flow finds a way to separate with unstable streamwise retardation in opposing flows replaced by quasi-stable transversal accelleration close to the surface before separation and in the swirling flow after separation. We see this phenomenon in the swirling flow in a bathtub drain, which is a stable configuration with transversal accelleration replacing the unstable opposing flow retardation of fully radial flow.

We refer to this quasi-stable pattern as *3d rotational separation*. This is a macroscopic phenomenon with the stagnation points spaced as widely as possibe. From macroscopic point of view the small skin friction of slightly viscous flow can be modeled with a *slip boundary condition* expressing vanishing skin fric-

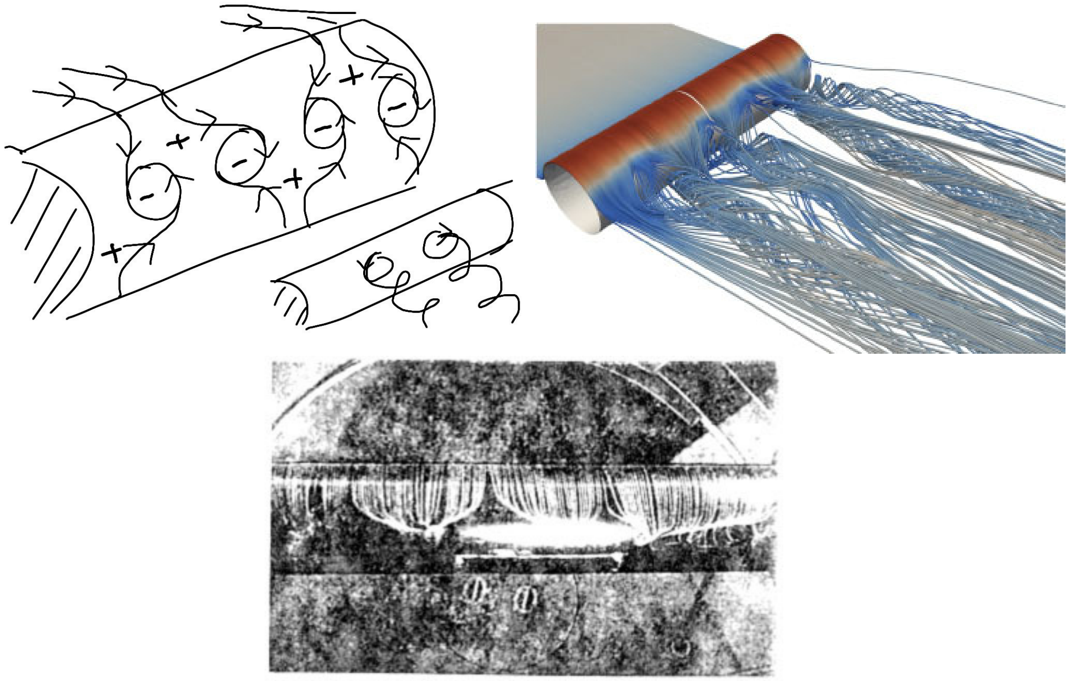


Figure 25.2: Quasi-stable 3d rotational separation from alternating high/low pressure: principle, computation and experiment

tion. We show that computational solution of Navier-Stokes equations with slip is possible at affordable cost, because with slip there are no boundary layers to resolve, which makes it possible to compute both drag and lift of a of a car, boat or airplane arbitrary shape without the quadrillions of mesh points for boundary layer resolution commonly believed to be required [43].

The single high pressure zone stretching along the stagnation line of potential flow around a circular cylinder (creating instability) in Fig. 25.1, is thus broken down into a pattern of high and low pressure zones by the development of low pressure vortical flow in Fig. 2, which allows the fluid to separate without unstable streamwise retardation in opposing flow. The so modified pressure creates drag of a bluff body and lift of a wing from the zero drag and lift of potential flow.

25.2 Resolution of D'Alembert's Paradox

Potential flow can be viewed as an approximate solution of the Navier-Stokes equations at high Reynolds number with a slip boundary condition, but potential flow is unphysical because both drag and lift are zero, as expressed in d'Alembert's paradox [104]. Inspection of potential flow shows unstable *irrotational separation* of retarding opposing flow, which is impossible to observe as a physical flow. D'Alembert's paradox is thus resolved by observing that potential flow with zero drag and lift is unstable [104] and thus unphysical, and not by the official resolution suggested by Prandtl stating that the unphysical feature is the slip boundary condition.

Although 3d rotational separation has a macroscopic features the flow is *turbulent* at separation in the sense that the dissipation in the flow is substantial even though the viscosity is very small, following the definition of turbulent flow in [103].

25.3 Main Result

We present evidence in the form of mathematical stability analysis and computation that high Reynolds number incompressible flow around a body moving through a fluid can be described as

- quasi-stable potential flow before separation,
- quasi-stable 3d rotational separation.

This scenario is also supported by observation presented in e.g. [121, 122, **?**] and our evidence thus consists of mathematical theory/computation and observation in strong accord.

We show that both drag and lift critically depend on the pressure distribution of 3d rotational separation. We remark that in the attaching flow in the front the flow is retarded by the body and not by opposing flows as in the back, which allows stable potential flow attachment. We show that drag and lift of a body of arbitrary shape can be accurately computed by solving the Navier-Stokes equations with slip.

The description and analysis of the crucial flow feature of separation presented here is fundamentally different from that of Prandtl, named the father of modern fluid mechanics, based on the idea that both drag and lift originate from a thin viscous boundary layer, where the flow speed relative to the body rapidly changes from the free stream speed to zero at the body surface corresponding to a no-slip boundary condition. Prandtl's scenario for separation, which has dominated 20th century fluid, can be described as *2d boundary layer no-slip separation*, to be compared with our entirely different scenario of *3d no-boundary layer slip separation*.

The unphysical aspect of Prandtl's scenario of separation is illuminated in [**?**]:

- *The passage from the familiar 2d to the mysterious 3d requires a complete reconsideration of concepts apparently obvious (separation and reattachment points, separated bubble, recirculation zone) but inappropriate and even dangerous to use in 3d flows.*

25.4 Navier-Stokes Equations with Slip by G2

We recall Navier-Stokes equations for incompressible flow with a slip boundary condition according to (**??**) with $\beta = 0$: Find the velocity $u = (u_1, u_2, u_3)$ and pressure p depending on $(x, t) \in \Omega \cup \Gamma \times I$, such that

$$\begin{array}{rcll} \dot{u} + (u \cdot \nabla)u + \nabla p - \nabla \cdot \sigma & = & f & \text{in } \Omega \times I, \\ \nabla \cdot u & = & 0 & \text{in } \Omega \times I, \\ u_n & = & g & \text{on } \Gamma \times I, \\ \sigma_s & = & 0 & \text{on } \Gamma \times I, \\ u(\cdot, 0) & = & u^0 & \text{in } \Omega, \end{array} \tag{25.1}$$

and that solutions can be computed using G2.

We have found that that G2 with slip is capable of modeling slightly viscous turbulent flow with $Re > 10^6$ of relevance in many applications in aero/hydro dynamics, including flying, sailing, boating and car racing, with hundred thousands of mesh points in simple geometry and millions in complex geometry, while according to state-of-the-art quadrillions is required [43]. This is because a friction-force/slip boundary condition can model a turbulent boundary layer, and interior turbulence does not have to be resolved to physical scales to capture mean-value outputs [103].

25.5 Stability Analysis by Linearization

The stability of a Navier-Stokes solution is expressed by the linearized equations

$$\begin{array}{rcll} \dot{v} + (u\cdot\nabla)v + (v\cdot\nabla)\bar{u} + \nabla q &=& f - \bar{f} & \text{in } \Omega\times I, \\ \nabla\cdot v &=& 0 & \text{in } \Omega\times I, \\ v\cdot n &=& g - \bar{g} & \text{on } \Gamma\times I, \\ v(\cdot,0) &=& u^0 - \bar{u}^0 & \text{in } \Omega, \end{array} \tag{25.2}$$

where (u, p) and $(\bar{u}, \bar{p})$ are two Euler solutions with slightly different data, and $(v, q) \equiv (u - \bar{u}, p - \bar{p})$. Formally, with u and $\bar{u}$ given, this is a linear convection-reaction problem for (v, q) with growth properties governed by the reaction term given by the 3×3 matrix $\nabla\bar{u}$. By the incompressiblity, the trace of $\nabla\bar{u}$ is zero, which shows that in general $\nabla\bar{u}$ has eigenvalues with real values of both signs, of the size of $|\nabla u|$ (with $|\cdot|$ some matrix norm), thus with at least one exponentially unstable eigenvalue, except in the neutrally stable case with purely imaginary eigenvalues, or in the non-normal case of degenerate eigenvalues representing parallel shear flow [103].

The linearized equations in velocity-pressure indicate that, as an effect of the reaction term $(v \cdot \nabla)\bar{u}$:

- streamwise retardation is exponentially unstable in velocity,
- transversal accelleration is neutrally stable,

where transversal signifies a direction orthogonal to the flow direction.

Additional stability information is obtained by applying the curl operator $\nabla\times$ to the momentum equation to give the vorticity equation

$$\dot{\omega} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \nabla\times f \quad \text{in } \Omega, \tag{25.3}$$

which is also a convection-reaction equation in the vorticity $\omega = \nabla \times u$ with coefficients depending on u, of the same form as the linearized equation (25.5), with a sign change of the reaction term. The vorticity is thus locally subject to exponential growth with exponent $|\nabla u|$:

- streamwise accelleration is exponentially unstable in streamwise vorticity.

We sum up as follows: The linearized equations (25.5) and (26.3) indicate exponential growth of perturbation of velocity in streamwise retardation and of streamwise vorticity in streamwise accelleration. We shall see in more detail below 3d rotational separation results from exponential instability of potential flow in retardation followed by vortex stretching in accelleration, with the retardation replaced by neutrally stable transversal accelleration.

Note that in classical analysis it is often argued that from the vorticity equation (26.3), it follows that vorticity cannot be generated starting from potential flow with zero vorticity and $f = 0$, which is *Kelvin's theorem*. But this is an incorrect conclusion, since perturbations of $\bar{f}$ of f with $\nabla \times \bar{f} \neq 0$ must be taken into account, even if $f = 0$. What you effectively see in computations is local exponential growth of vorticity on the body surface in rear retardation and by vortex stretching in accelleration, even if $f = 0$, which is a main route of instability to turbulence as well as separation.

25.6 Instability of 2d Irrotational Separation

We now analyze the stability of 2d irrotational separation considered by Planck in the following model of the potential flow around a circular cylinder studied in more detail below: $u(x) = (x_1, -x_2, 0)$ in the half-plane $\{x_1 > 0\}$ with stagnation along the line $(0, 0, x_3)$ and

$$\frac{\partial u_1}{\partial x_1} = 1 \quad \text{and} \quad \frac{\partial u_2}{\partial x_2} = -1, \tag{25.4}$$

expressing that the fluid is *squeezed* by *retardation* in the x_2-direction and *accelleration* in the x_1-direction. We first focus on the retardation with the main stability feature of (25.5) captured in the following simplified version of the v_2-equation of (25.5), assuming x_1 and x_2 are small,

$$\dot{v}_2 - v_2 = f_2,$$

where we assume $f_2 = f_2(x_3)$ to be an oscillating perturbation depending on x_3 of a certain wave length δ and amplitude h, for example $f_2(x_3) = h\sin(2\pi x_3/\delta)$, expecting the amplitude to decrease with the wave length. We find, assuming $v_2(0,x) = 0$, that

$$v_2(t, x_3) = (\exp(t) - 1)f_2(x_3).$$

We next turn to the accelleration and then focus on the ω_1-vorticity equation, for x_2 small and $x_1 \geq \bar{x}_1 > 0$ with $\bar{x}_1$ small, approximated by

$$\dot{\omega}_1 + x_1\frac{\partial \omega_1}{\partial x_1} - \omega_1 = 0,$$

with the "inflow boundary condition"

$$\omega_1(\bar{x}_1, x_2, x_3) = \frac{\partial v_2}{\partial x_3} = (\exp(t) - 1)\frac{\partial f_2}{\partial x_3}.$$

The equation for ω_1 thus exhibits exponential growth, which is combined with exponential growth of the "inflow condition". We can see these features in principle and computational simulation in Fig. **??** showing how opposing flows at separation generate a pattern of alternating surface vortices from pushes of fluid up/down, which act as initial conditions for vorticity stretching into the fluid generating counter-rotating low-pressure tubes of streamwise vorticity.

The above model study can be extended to the full linearized equations linearized at $u(x) = (x_1, -x_2, 0)$:

$$\begin{array}{rcl} Dv_1 + v_1 & = & -\frac{\partial q}{\partial x_1}, \\ Dv_2 - v_2 & = & -\frac{\partial q}{\partial x_2} + f_2(x_3), \\ Dv_3 & = & -\frac{\partial q}{\partial x_3}, \\ \nabla \cdot v & = & 0 \end{array} \tag{25.5}$$

where $Dv = \dot{v} + u \cdot \nabla v$ is the convective derivative with velocity u and $f_2(x_3)$ as before. We here need to show that the force perturbation $f_2(x_3)$ will not get cancelled by the pressure term $-\frac{\partial q}{\partial x_2}$ in which case the exponential growth of v_2 would get cancelled. Now $f_2(x_3)$ will induce a variation of v_2 in the x_3 direction, but this variation does not upset the incompressibility since it involves the variation in x_2. Thus, there is no reason for the pressure q to compensate for the force perturbation f_2 and thus exponential growth of v_2 is secured.

We thus find streamwise vorticity generated by a force perturbation oscillating in the x_3 direction, which in the retardation of the flow in the x_2-direction creates exponentially increasing vorticity in the x_1-direction, which acts as inflow to the ω_1-vorticity equation with exponential growth by vortex stretching. Thus, we find exponential growth at rear separation in both the retardation in the x_2-direction and the accelleration in the x_1 direction, as a result of the squeezing expressed by (25.4).

Since the combined exponential growth is independent of δ, it follows that large-scale perturbations with large amplitude have largest growth, which is also seen in computations with δ the distance between streamwise rolls as seen in Fig. 26.3 which does not seem to decrease with decreasing h. The perturbed flow with swirling separation is large scale phenomenon, which we show below is more stable than potential flow.

The corresponding pressure perturbation changes the high pressure at separation of potential flow into a zig-zag alternating more stable pattern of high and low pressure with high pressure zones deviating opposing flow into non-opposing streaks which are captured by low pressure to form rolls of streamwise vorticity allowing the flow to spiral away from the body. This is similar to the vortex formed in a bathtub rain.

Notice that in attachment in the front the retardation does not come from opposing flows but from the solid body, and the zone of exponential growth of ω_2 is short, resulting in much smaller perturbation growth than at rear separation.

We shall see that the tubes of low-pressure streamwise vorticity change the normal pressure gradient to allow separation without unstable retardation, but the price is generation of drag by negative pressure inside the vortex tubes as a "cost of separation".

25.7 Quasi-Stable Rotational 3d Separation

We discover in computation and experiment that the rotational 3d separtion pattern just detected as the most unstable mode of 2d, represents a quasi-stable flow with unstable retardation in opposing flows replaced by transversal acceleration.

As a model of flow with transversal accelleration we consider the potential velocity $u = (0, x_3, -x_2)$ of a constant rotation in the x_1-direction, with corresponding linearized equations linearized problem

$$\dot{v}_1 = 0, \quad \dot{v}_2 + v_3 = 0, \quad \dot{v}_3 - v_2 = 0, \tag{25.6}$$

which model a neutrally stable harmonic oscillator without exponential growth corresponding to imaginary eigenvalues of ∇u.

Further, shear flow may represented by $(x_2, 0, 0)$, which is marginal unstable with linear perturbation growth from degenerate zero eigenvalues of ∇u, as analyzed in detail in [103].

25.8 Quasi-Stable Potential Flow Attachment

The above analysis also shows that potential flow attachment, even though it involves streamwise retardation, is quasi-stable. This is because the initial perturbation f_2 in the above analysis is forced to be zero by the slip boundary condition requiring the normal velocity to vanish. In short, potential flow attachment is stable because the flow is retarded by the solid body and not by opposing flows as in separation.

This argument further shows that a flow retarded by a high pressure zone is quasi-stable in approach because it is similar to attachment.

Chapter 26

Basic Cases

26.1 Circular Cylinder

We consider the flow around a around a long circular cylinder of unit radius with axis along the x_3-axis in $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$, assuming the flow velocity is $(1, 0, 0)$ at infinity.

26.1.1 Unstable Unphysical Potential Flow

Potential flow as inviscid, irrotational, incompressible stationary flow, is given in polar coordinates (r, θ) in a plane orthogonal to the cylinder axis by the potential function, see Fig. 26.1,

$$\varphi(r,\theta) = (r + \frac{1}{r})\cos(\theta)$$

with corresponding velocity components

$$u_r \equiv \frac{\partial\varphi}{\partial r} = (1 - \frac{1}{r^2})\cos(\theta), \quad u_s \equiv \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -(1 + \frac{1}{r^2})\sin(\theta)$$

with streamlines being level lines of the conjugate potential function

$$\psi \equiv (r - \frac{1}{r})\sin(\theta).$$

Potential flow is constant in the direction of the cylinder axis with velocity $(u_r, u_s) = (1, 0)$ for r large, is fully symmetric with zero drag/lift, attaches and separates at the lines of stagnation $(r, \theta) = (1, \pi)$ in the front and $(r, \theta) = (1, 0)$ in the

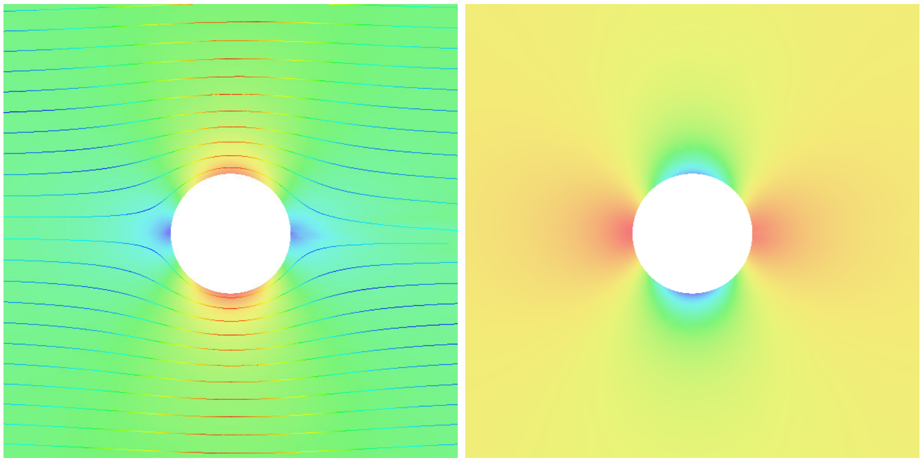


Figure 26.1: Potential flow past a circular cylinder: fully symmetric velocity (left) and pressure (right).

back. Potential flow shows exponentially unstable 2d irrotational separation but quasi-stable 2d attachment. Potential flow thus represents physical flow before separation but not in separation and after separation.

By Bernouilli's principle the pressure is given by

$$p = -\frac{1}{2r^4} + \frac{1}{r^2}\cos(2\theta)$$

when normalized to vanish at infinity. We compute

$$\frac{\partial p}{\partial \theta} = -\frac{2}{r^2}\sin(2\theta)), \quad \frac{\partial p}{\partial r} = \frac{2}{r^3}(\frac{1}{r^2} - \cos(2\theta)),$$

and discover an adverse pressure gradient in the back. Further, the normal pressure gradient on the boundary

$$\frac{\partial p}{\partial r} = 4\sin^2(\theta) \geq 0$$

is precisely the force required to accelerate fluid particles with speed $2|\sin(\theta)|$ to follow the circular boundary without separation, by satisfying the condition of non-separation on a curve with curvature R

$$\frac{\partial p}{\partial n} = \frac{U^2}{R}. \tag{26.1}$$

We note, coupling to the above discussion relating to (**??**), that $\frac{\partial u_s}{\partial r} = \frac{2}{r^3}\sin(\theta) = 2$ at the crest. We further compute

$$\frac{\partial \psi}{\partial r} = \frac{1}{r^2}\sin(\theta)$$

which shows that fluid particles decrease their distance to the boundary in front of the cylinder and increase their distance in the rear, but the flow only separates at rear stagnation.

26.1.2 Quasi-Stable Physical Turbulent Flow

Solving Navier-Stokes equations with very small viscsoity and slip boundary condition by G2 we find the a a flow initialized as potential flow develops into a turbulent solution with rotational separation as identified above, in shown in Fig. 26.2 and 26.3.

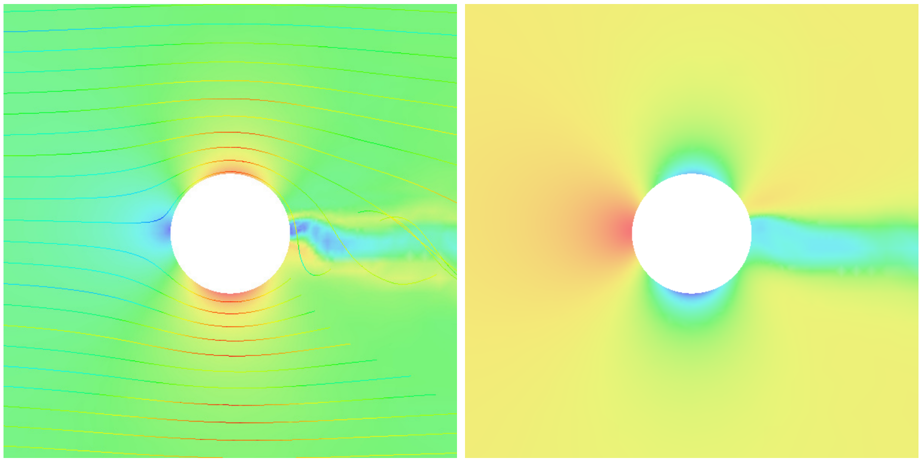


Figure 26.2: Turbulent flow past a cylinder; velocity (left) and pressure (right). Notice the low pressure wake of strong streamwise vorticity generating drag.

26.2 NACA0012 Trailing Edge Separation

The separation at the trailing edge of a wing is similar to that of a circular cylinder, as shown in Fig. 26.2 for a NACA012 wing at 5 degrees angle of attack.

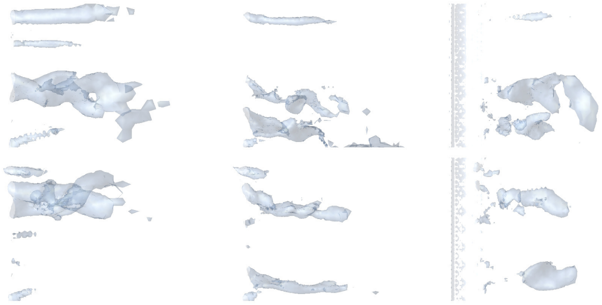


Figure 26.3: Levels surfaces of strong vorticity in EG2 solution: streamwise $|\omega_1|$ (left) and transversal $|\omega_2|$ (middle) and $|\omega_3|$ (right), at two times $t_1 < t_2$ (upper, lower), in the x_1x_3-plane.

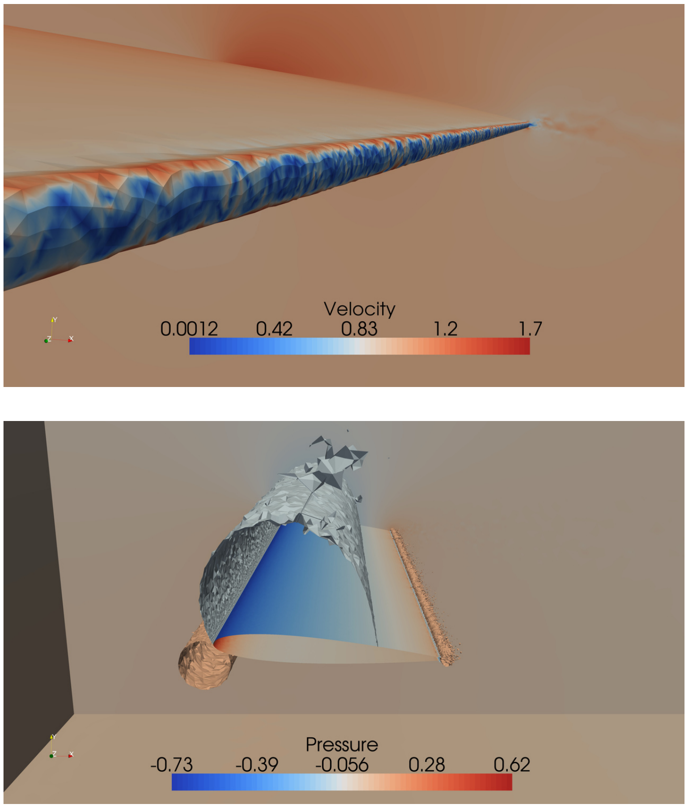


Figure 26.4: Velocity, pressure and vorticity at trailing edge separation for NACA0012 wing. Notice the zig-zag pattern of the velocity.

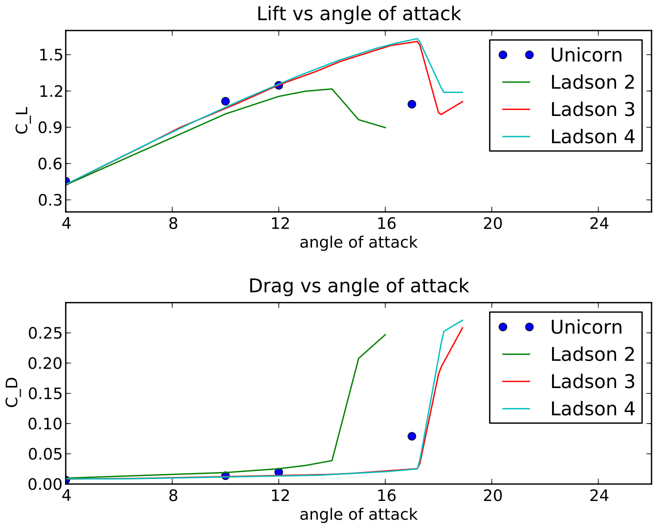


Figure 26.5: Evidence that drag and lift of a wing can be computed by solving the Navier-Stokes equations with slip without resolving any boundary layers.

26.3 Accuracte Drag and Lift without Boundary Layer

We compare in Fig. 26.6 drag and lift of a long NACA0012 wing for different angles of attack including stall computed by solving the of Navier-Stokes equations with slip using Unicorn **??**, with different experiments and notice good agreement. We conclude that drag and lift are computable without resolving and boundary layers.

26.4 Sphere

Potential flow around a sphere is exponentially unstable at its point of stagnation at separation and develops a quasi-stable separation pattern of four counterrotating rolls of streamwise vorticity as shown in Fig. 26.4.

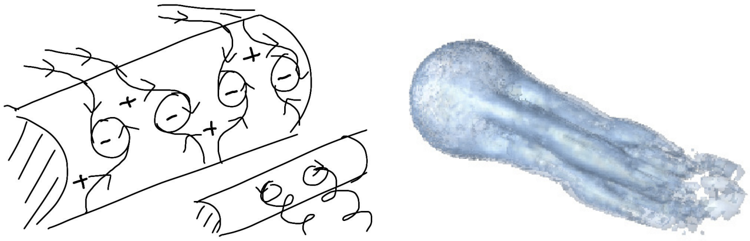


Figure 26.6: Pattern of exponential instability of potential flow around a sphere at point of stagnation forming four counterrotating rolls of streamwise vorticity, shown in computation.

26.5 Hill

In Fig. 26.7 we show turbulent Euler flow over a hill with separation after the crest by again the mechanism of tangential separation through generation of surface vorticity.

26.6 Flat Plate

The experience reported above suggests the following scenario for separation into a turbulent boundary layer over a flat plate as a representation of a smooth boundary: (i) Rolls of streamwise vorticity are formed by non-modal linear perturbation growth referred to as the *Taylor Görtler mechanism* in [103]. (ii) The rolls create opposing transversal flows (as in the back of cylinder), which generate surface vorticity which is stretched into the fluid while being bent into to streamwise direction, as evidenced in e.g. [29, 30].

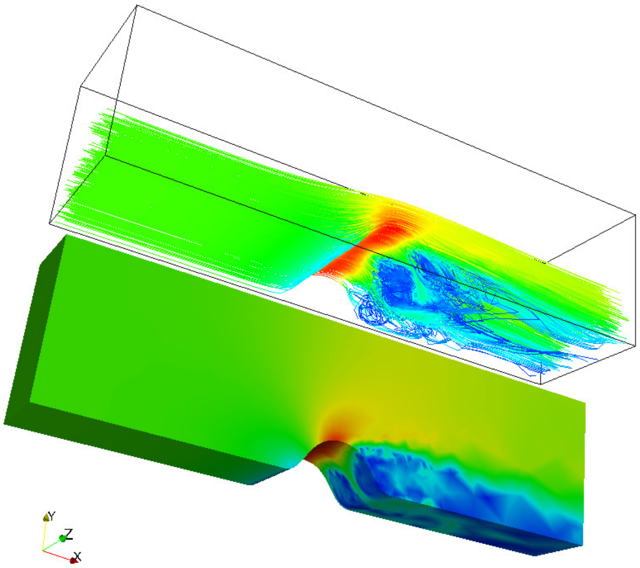


Figure 26.7: Separation after crest of hill by surface vorticity from opposing flow.

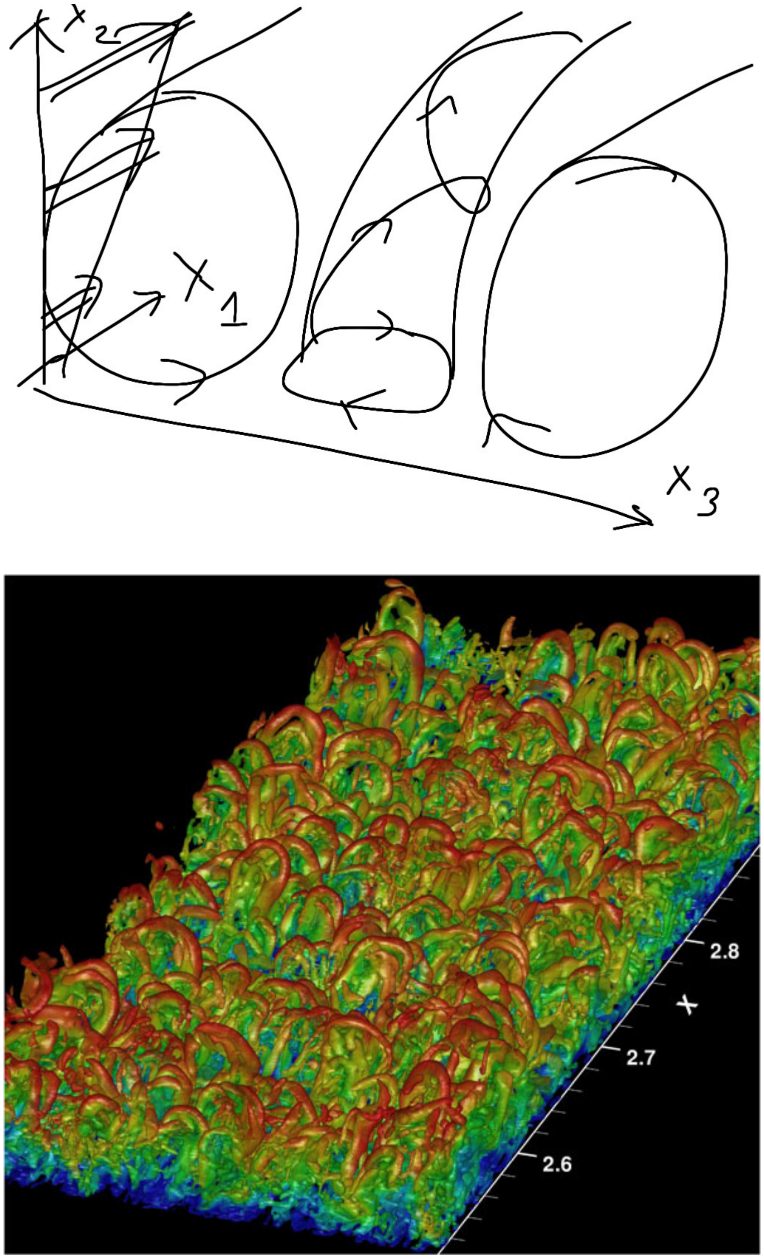


Figure 26.8: Shear flow over flat plate generates x_1 vorticity which generates secondary transversal opposing flow which generates rolls of x_2-vorticity attaching to the plate and bending into the flow, like a forest of sea tulips attaching to the sea bottom.

Chapter 27

Energy Estimate

The standard *energy estimate* for (20.1) is obtained by multiplying the momentum equation

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nabla \cdot \sigma - f = 0,$$

with u and integrating in space and time, to get in the case $f = 0$ and $g = 0$,

$$\int_0^t \int_\Omega R_\nu(u,p) \cdot u \, dxdt = D_\nu(u;t) + B_\beta(u;t) \tag{27.1}$$

where

$$R_\nu(u,p) = \dot{u} + (u \cdot \nabla)u + \nabla p$$

is the *Euler residual* for a given solution (u,p) with $\nu > 0$,

$$D_\nu(u;t) = \int_0^t \int_\Omega \nu |\epsilon(u(\bar{t},x))|^2 dxd\bar{t}$$

is the *internal turbulent viscous dissipation*, and

$$B_\beta(u;t) = \int_0^t \int_\Gamma \beta |u_s(\bar{t},x)|^2 dxd\bar{t}$$

is the *boundary turbulent viscous dissipation*, from which follows by standard manipulations of the left hand side of (27.1),

$$K_\nu(u;t) + D_\nu(u;t) + B_\beta(u;t) = K(u^0), \quad t > 0, \tag{27.2}$$

where

$$K_\nu(u;t) = \frac{1}{2} \int_\Omega |u(t,x)|^2 dx.$$

This estimate shows a balance of the *kinetic energy* $K(u;t)$ and the *turbulent viscous dissipation* $D_\nu(u;t) + B_\beta(u;t)$, with any loss in kinetic energy appearing as viscous dissipation, and vice versa. In particular,

$$D_\nu(u;t) + B_\beta(u;t) \leq K(u^0),$$

and thus the viscous dissipation is bounded (if $f = 0$ and $g = 0$).

Turbulent solutions of (20.1) are characterized by *substantial internal turbulent dissipation*, that is (for t bounded away from zero),

$$D(t) \equiv \lim_{\nu \to 0} D(u_\nu;t) >> 0, \tag{27.3}$$

which is *Kolmogorov's conjecture* [97]. On the other hand, the *skin friction dissipation* decreases with decreasing friction

$$\lim_{\nu \to 0} B_\beta(u;t) = 0, \tag{27.4}$$

since $\beta \sim \nu^{0.2}$ tends to zero with the viscosity ν and the tangential velocity u_s approaches the (bounded) free-stream velocity. We thus find evidence that the interior turbulent dissipation dominates the skin friction dissipation, which supports the use of slip as a model of a turbulent boundray layer, but which is not in accordance with Prandtl's (unproven) conjecture that substantial drag and turbulent dissipation originates from the boundary layer.

Kolmogorov's conjecture (27.3) is consistent with

$$\|\nabla u\|_0 \sim \frac{1}{\sqrt{\nu}}, \quad \|R_\nu(u,p)\|_0 \sim \frac{1}{\sqrt{\nu}}, \tag{27.5}$$

where $\|\cdot\|_0$ denotes the $L_2(Q)$-norm with $Q = \Omega \times I$. On the other hand, it follows by standard arguments from (27.2) that

$$\|R_\nu(u,p)\|_{-1} \leq \sqrt{\nu}, \tag{27.6}$$

where $\|\cdot\|_{-1}$ is the norm in $L_2(I; H^{-1}(\Omega))$. Kolmogorov thus conjectures that the Euler residual $R_\nu(u,p)$ for small ν is strongly (in L_2) large, while being small weakly (in H^{-1}).

Altogether, we understand that the resolution of d'Alembert's paradox of explaining substantial drag from vanishing viscosity, consists of realizing that the internal turbulent dissipation D can be positive under vanishing viscosity, while the skin friction dissipation B will vanish. In contradiction to Prandtl, we conclude that drag does not result from boundary layer effects, but from internal turbulent dissipation, originating from instability at separation.

Chapter 28

G2 Computational Solution

We show in [103, 102, 104] that the Navier-Stokes equations (20.1) can be solved by G2 producing turbulent solutions characterized by substantial turbulent dissipation from the least squares stabilization acting as an automatic turbulence model, reflecting that the Euler residual cannot be made pointwise small in turbulent regions. G2 has a posteriori error control based on duality and shows output uniqueness in mean-values such as lift and drag [103, 99, 100]

We find that G2 with slip is capable of modeling slightly viscous turbulent flow with $Re > 10^6$ of relevance in many applications in aero/hydro dynamics, including flying, sailing, boating and car racing, with hundred thousands of mesh points in simple geometry and millions in complex geometry, while according to state-of-the-art quadrillions is required [59]. This is because a friction-force/slip boundary condition can model a turbulent boundary layer, and interior turbulence does not have to be resolved to physical scales to capture mean-value outputs [103].

The idea of circumventing boundary layer resolution by relaxing no-slip boundary conditions introduced in [99, 103], was used in [115, 26] in the form of weak satisfaction of no-slip, which however misses the main point of using a force condition instead of a velocity condition in a model of a turbulent boundary layer.

A G2 solution (U, P) on a mesh with local mesh size $h(x,t)$ according to [103], satisfies the following energy estimate (with $f = 0$, $g = 0$ and $\beta = 0$):

$$K(U(t)) + D_h(U;t) = K(u^0), \tag{28.1}$$

where

$$D_h(U;t) = \int_0^t \int_\Omega h|R_h(U,P)|^2 \, dxdt, \tag{28.2}$$

is an analog of $D_\nu(u;t)$ with $h \sim \nu$, where $R_h(U,P)$ is the Euler residual of (U,P). We see that the G2 turbulent viscosity $D_h(U;t)$ arises from penalization of a non-zero Euler residual $R_h(U,P)$ with the penalty directly connecting to the violation (according the theory of criminology). A turbulent solution is characterized by substantial dissipation $D_h(U;t)$ with $\|R_h(U,P)\|_0 \sim h^{-1/2}$, and

$$\|R_h(U,P)\|_{-1} \leq \sqrt{h} \tag{28.3}$$

in accordance with (27.5) and (27.6).

28.1 Wellposedness of Mean-Value Outputs

Let $M(v) = \int_Q v\psi\, dxdt$ be a *mean-value output* of a velocity v defined by a smooth weight-function $\psi(x,t)$, and let (u,p) and (U,P) be two G2-solutions on two meshes with maximal mesh size h. Let (φ,θ) be the solution to the *dual linearized problem*

$$\begin{array}{rcll} -\dot\varphi - (u\cdot\nabla)\varphi + \nabla U^\top\varphi + \nabla\theta &=& \psi & \text{in } \Omega\times I,\\ \nabla\cdot\varphi &=& 0 & \text{in } \Omega\times I,\\ \varphi\cdot n &=& g & \text{on } \Gamma\times I,\\ \varphi(\cdot,T) &=& 0 & \text{in } \Omega, \end{array} \tag{28.4}$$

where $\top$ denotes transpose. Multiplying the first equation by $u-U$ and integrating by parts, we obtain the following output error representation [103]:

$$M(u) - M(U) = \int_Q (R_h(u,p) - R_h(U,P))\cdot\varphi\, dxdt \tag{28.5}$$

where for simplicity the dissipative terms are here omitted, from which follows the a posteriori error estimate:

$$|M(u) - M(U)| \leq S(\|R_h(u,p)\|_{-1} + \|R_h(U,P)\|_{-1}), \tag{28.6}$$

where the stability factor

$$S = S(u,U,M) = S(u,U) = \|\varphi\|_{H^1(Q)}. \tag{28.7}$$

In [103] we present a variety of evidence, obtained by computational solution of the dual problem, that for global mean-value outputs such as drag and lift,

$S << 1/\sqrt{h}$, while $\|R_h\|_{-1} \sim \sqrt{h}$, allowing computation of of drag/lift with a posteriori error control of the output within a tolerance of a few percent. In short, mean-value outputs such as lift and drag are wellposed and thus physically meaningful.

We explain in [103] the crucial fact that $S << 1/\sqrt{h}$, heuristically as an effect of *cancellation* of rapidly oscillating reaction coefficients of turbulent solutions combined with smooth data in the dual problem for mean-value outputs. In smooth potential flow there is no cancellation, which explains why zero lift/drag cannot be observed in physical flows.

Part V

History

Chapter 29

Aristotele

> Probable impossibilities are to be preferred to improbable possibilities. (Aristotle)
>
> In all things of nature there is something of the marvelous. (Aristotle)

We have seen that the secret of flying is how to generate large lift by the motion of a wing through air at the expense of small drag. Approaching this problem we face the general problem of *motion*, seriously addressed already by the Greek philosopher *Zeno* in his famous paradox about the arrow, which in every single moment along its path seems to be frozen into immobility, but yet effectively is moving. Today we are familiar with many forms of motion and most people would probably say that Zeno's paradox must have been resolved since long, although they would not be able to account for the details of the resolution.

However, the true nature of e.g. the motion of light through vacuum still seems to be hidden to us, while the motion through a gas/fluid like air or water can be approached following an idea presented already by *Aristotle* in the 4th century BC known as *antiperistasis*. Aristotle states in his *Physics* that that a body in motion through air is pushed from behind by the stream of air around the body contracting in the rear after having been expanded in the front. This is like the *peristaltic* muscle contractions that propels foodstuffs distally through the esophagus and intestines, which is like the squeezing of an object through a lubricated elastic tube by the combined action of the object expanding the tube in the front and the tube contracting in the rear of the object, as expressed in the words of Aristotle:

- *Thirdly, in point of fact things that are thrown move though that which gave them their impulse is not touching them, either by reason of mutual replacement, as some maintain, or because the air that has been pushed pushes* *them with a movement quicker than the natural locomotion of the projectile wherewith it moves to its proper place. But in a void none of these things can take place; the only way anything can move is by riding on something else.*

- *Fourthly, no one could say why a thing once set in motion should stop anywhere; for why should it stop here rather than here? So that a thing will either be at rest or must be moved ad infinitum, unless something stronger than it impedes it.*
- *Fifthly, things are now thought to move into the void because it yields; but in a void this quality is present equally everywhere, so that things should move in all directions.*

Of course, we say today that according to Newton's 2nd law, a body will continue in rectilinear motion at constant speed unless acted upon by some force, while to Aristotle sustained motion would require a force pushing from behind. Nevertheless, we will find that there is something in Aristotle's antiperistasis which correctly describes an important aspect of motion through air, if not through vacuum, but you cannot fly in vaccum...

29.1 Liberation from Aristotle

The renewal of learning in Europe, that began with 12th century Scholasticism, came to an end about the time of the Black Death, but the Northern Renaissance (in contrast to th Italian) showed a decisive shift in focus from Aristoteleian natural philosophy to chemistry and the biological sciences. Thus modern science in Europe was resumed in a period of great upheaval: the Protestant Reformation and Catholic Counter-Reformation; the discovery of the Americas by Christopher Columbus; the Fall of Constantinople; but also the re-discovery of Aristotle during the Scholastic period presaged large social and political changes. Thus, a suitable environment was created in which it became possible to question scientific doctrine, in much the same way that Martin Luther and John Calvin questioned religious doctrine. The works of Ptolemy (astronomy) and Galen (medicine) were found not always to match everyday observations.

The willingness to question previously held truths and search for new answers opened the *Scientific Revolution* by Copernicus' *De Revolutionibus* in 1543 stating that the Earth moved around the Sun, followed by Newton's *Principia Mathematica* in 1687.

Chapter 30

Medieval Islamic Physics of Motion

> When hearing something unusual, do not preemptively reject it, for that would be folly. Indeed, horrible things may be true, and familiar and praised things may prove to be lies. Truth is truth unto itself, not because [many] people say it is. (Ibn Al-Nafis, 1213-1288 A.D.)

> We start by observing reality we try to select solid (unchanging) observations that are not affected by how we perceive (measure) them. We then proceed by increasing our research and measurement, subjecting premises to criticism, and being cautious in drawing conclusions In all we do, our purpose should be balanced not arbitrary, the search for truth, not support of opinions...Hopefully, by following this method, this road to the truth that we can be confident in, we shall arrive to our objective, where we feel certain that we have, by criticism and caution, removed discord and suspicion...Yet we are but human, subject to human frailties, against which we must fight with all our human might. God help us in all our endeavors. (Ibn Al-Haytham)

> The knowledge of anything, since all things have causes, is not acquired or complete unless it is known by its causes. (Avicenna)

Medieval Islamic developments in mechanics prepared for the liberation of science from Christian Scholasticism following Aristotle's legacy, through the new mechanics of Galileo and Newton leading into the Enlightment and modern Europe. We recall some early Islamic scientists questioning Aristotle, and preparing for human flight...

30.1 Avicenna

Avicenna (980-1037) a foremost Persian polymath developed an elaborate theory of motion, in which he made a distinction between the inclination and force of a projectile, and concluded that motion was a result of an inclination (mayl) transferred to the projectile by the thrower, and that projectile motion in a vacuum would not cease. He viewed inclination as a permanent force whose effect is dissipated by external forces such as air resistance. He also developed the concept of momentum, referring to impetus as being proportional to weight times velocity. His theory of motion was also consistent with the concept of inertia in classical mechanics, and later formed the basis of Jean Buridan's theory of impetus and exerted an influence on the work of Galileo Galilei.

30.2 Abu'l-Barakat

Hibat Allah Abu'l-Barakat al-Baghdaadi (1080-1165) wrote a critique of Aristotelian physics where he was the first to negate Aristotle's idea that a constant force produces uniform motion, as he realized that a force applied continuously produces acceleration as an early foreshadowing of Newton's second law of motion. He described acceleration as the rate of change of velocity and modified Avicenna's view on projectile motion stating that the mover imparts a violent inclination on the moved and that this diminishes as the moving object distances itself from the mover. Abu'l-Barakat also suggested that motion is relative.

30.3 Biruni

Another prominent Persian polymath, *Abu Rayan Biruni*, engaged in a written debate with Avicenna, with Biruni criticizing the Peripatetic school for its adherence to Aristotelian physics and natural philosophy preserved in a book entitled *al-As'ila wal-Ajwiba* (Questions and Answers). al-Biruni attacks Aristotle's theories on physics and cosmology, and questions almost all of the fundamental Aristotelian physical axioms. He rejects the notion that heavenly bodies have an inherent nature and asserts that their *"motion could very well be compulsory"* and maintains that *"there is no observable evidence that rules out the possibility of vacuum"*; and states that there is no inherent reason why planetary orbits must be circular and cannot be elliptical. He also argues that *"the metaphysical ax-*

ioms on which philosophers build their physical theories do not constitute valid evidence for the mathematical astronomer", which marks the first real distinction between the vocations of the *philosopher-metaphysician* (like Aristotle and Avicenna) and that of the *mathematician-scientist* (al-Biruni himsel). In contrast to the philosophers, the only evidence that al-Biruni considered reliable were either mathematical or empirical evidence, and his systematic application of rigorous mathematical reasoning later led to the mathematization of Islamic astronomy and the mathematization of nature.

30.4 Biruni's Questions

Biruni began the debate by asking Avicenna eighteen questions, ten of which were criticisms of Aristotle's *On the Heavens*, which represents an early example of the scientific method of questioning basic postulates which we seek to use in thi book.

The first question criticized the Aristotelian theory of gravity for denying the existence of levity or gravity in the celestial spheres, and the Aristotelian notion of circular motion being an innate property of the heavenly bodies.

Biruni's second question criticizes Aristotle's over-reliance on more ancient views concerning the heavens, while the third criticizes the Aristotelian view that space has only six directions. The fourth question deals with the continuity and discontinuity of physical bodies, while the fifth criticizes the Peripatetic denial of the possibility of there existing another world completely different from the world known to them. In his sixth question, Biruni rejects Aristotle's view on the celestial spheres having circular orbits rather than elliptic orbits. In his seventh question, he rejects Aristotle's notion that the motion of the heavens begins from the right side and from the east, while his eighth question concerns Aristotle's view on the fire element being spherical.

The ninth question concerns the movement of heat, and the tenth question concerns the transformation of elements. The eleventh question concerns the burning of bodies by radiation reflecting off a flask filled with water, and the twelfth concerns the natural tendency of the classical elements in their upward and downward movements.

The thirteenth question deals with vision, while the fourteenth concerns habitation on different parts of Earth. His fifteenth question asks how two opposite squares in a square divided into four can be tangential, while the sixteenth question concerns vacuum. His seventeenth question asks *"if things expand upon heating and contract upon cooling, why does a flask filled with water break when water* *freezes in it?"* His eighteenth and final question concerns the observable phenomenon of ice floating on water. After Avicenna responded to the questions, Biruni was unsatisfied with some of the answers and wrote back commenting on them.

30.5 Ibn al-Haytham

Ibn al-Haytham (965-1039) discussed the theory of attraction between masses, and it seems that he was aware of the magnitude of acceleration due to gravity and he stated that the heavenly bodies *"were accountable to the laws of physics"*. Ibn al-Haytham also enunciated the law of inertia, later known as Newton's first law of motion, when he stated that a body moves perpetually unless an external force stops it or changes its direction of motion. He also developed the concept of momentum, though he did not quantify this concept mathematically. Nobel Prize winning physicist Abdus Salam wrote the following on Ibn al-Haytham: Ibn-al-Haitham was one of the greatest physicists of all time. He made experimental contributions of the highest order in optics. He enunciated that a ray of light, in passing through a medium, takes the path which is the easier and "quicker". In this he was anticipating Fermat's Principle of Least Time by many centuries. He enunciated the law of inertia, later to become Newton's first law of motion. Part V of Roger Bacon's *Opus Majus* is practically an annotation to Ibn al Haitham's Optics.

30.6 Others

Ibn Bajjah (d. 1138) argued that there is always a reaction force for every force exerted, connecting to Newton's 3rd law, though he did not refer to the reaction force as being equal to the exerted force, which had an important influence on later physicists like Galileo. *Averroes* (1126-1198) defined and measured force as *"the rate at which work is done in changing the kinetic condition of a material body'* and correctly argued *"that the effect and measure of force is change in the kinetic condition of a materially resistant mass."* In the 13th century, *Nasir al-Din al-Tusi* stated an early version of the law of conservation of mass, noting that a body of matter is able to change, but is not able to disappear. In the early 16th century, al-Birjandi developed a hypothesis similar to Galileo's notion of *"circular inertia.*

> At night I would return home, set out a lamp before me, and devote myself

to reading and writing. Whenever sleep overcame me or I became conscious of weakening, I would turn aside to drink a cup of wine, so that my strength would return to me. Then I would return to reading. And whenever sleep seized me I would see those very problems in my dream; and many questions became clear to me in my sleep. I continued in this until all of the sciences were deeply rooted within me and I understood them as is humanly possible. Everything which I knew at the time is just as I know it now; I have not added anything to it to this day. Thus I mastered the logical, natural, and mathematical sciences, and I had now reached the science. (Avicenna)

Chapter 31

Leonardo da Vinci

The noblest pleasure is the joy of understanding. (da Vinci)

Although nature commences with reason and ends in experience it is necessary for us to do the opposite, that is to commence with experience and from this to proceed to investigate the reason...He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast. (da Vinci)

For once you have tasted flight you will walk the earth with your eyes turned skywards, for there you have been and there you will long to return. (da Vinci)

Life is pretty simple: You do some stuff. Most fails. Some works. You do more of what works. If it works big, others quickly copy it. Then you do something else. The trick is the doing something else. (da Vinci)

Nothing strengthens authority so much as silence...You do ill if you praise, but worse if you censure, what you do not understand...There are three classes of people: those who see, those who see when they are shown, those who do not see...And many have made a trade of delusions and false miracles, deceiving the stupid multitude...Beware of the teaching of these speculators, because their reasoning is not confirmed by experience. (da Vinci)

31.1 The Polymath

Leonardo da Vinci (1452-1519) is the greatest polymath, universal genious, homo universale or renaissance man all times, with remarkable achievements as a scientist, mathematician, engineer, inventor, anatomist, painter, sculptor, architect, botanist, musician and writer.

Born as the illegitimate son of a notary, Piero da Vinci, and a peasant woman, Caterina, at Vinci in the region of Florence, Leonardo was educated in the studio of the renowned Florentine painter, Verrocchio. Much of his earlier working life was spent in the service of Ludovico il Moro in Milan. He later worked in Rome, Bologna and Venice and spent his last years in France, at the home awarded him by King Francois I.

As an artist Leonardo created the most famous, most reproduced and most parodied portrait and religious painting of all time: Mona Lisa and The Last Supper. As a scientist, he greatly advanced the state of knowledge in the fields of anatomy, civil engineering, optics, and hydrodynamics. As an engineer he conceptualised a helicopter, a tank, concentrated solar power, a calculator, the double hull and outlined a rudimentary theory of plate tectonics.

31.2 The Notebooks

Da Vinci recorded his discoveries in journals or *Notebooks* mostly written in mirror-image cursive, probably for practical expediency because Leonardo was left-handed, rather than for reasons of secrecy because it appears they were intended for publication. Although his language was clear and expressive, Leonardo preferred illustration to the written word stating, in the spirit of modern pedagogics:

- *The more detail you write concerning it the more you will confuse the reader*.

It is believed that there were at least 50 notebooks left in the hands of da Vinci's pupil Francesco Melzi at the master's death in 1519, of which 28 remains, but they were virtually unknown during his life-time and remained hidden for over two centuries. His wonderful ideas were forgotten; his inventions were not tested and built for hundreds of years. Dan Brown's best-seller *The Da Vinci Code* have stimulated renewed interest in da Vinci and his complex and inquiring intelligence. Today we can recoognize as an early precursor of an entire lineage oc scientists and philosophers whose central focus was the nature of organic form [4].

Leonardo planned to treat four major themes: the science of painting, architecture, the elements of mechanics, and a general work on human anatomy. To these themes were eventually added notes on his studies of botany, geology, flight, and hydrology. His intention was to combine all his investigations with a unified world view:

- *Plan of Book 15: First write of all water, in each of its motions; then describe all its bottoms and their various materials, always referring to the propositions concerning the said waters; and let the order be good, for otherwise the work will be confused. Describe all the forms taken by water from its greatest to its smallest wave, and their causes.*

Da Vinci made impressive and comprehensive investigations into aerodynamics collected into his *Codex on the Flight of Birds* from 1505, and designed a large variety of *ornithopters* for muscle-powered human flight using flapping wings. After extensive testing da Vinci concluded that even if both arms and legs got involved through elaborate mechanics, human power was insufficient for flapping flight, but during his last years in Florence he began to experiment with designs of flying machines that had fixed wings, not unlike modern hang-gliders.

31.3 The Scientist

Da Vinci observes the dynamics of the physical world with mountains, rivers, plants and the human body in ceaseless movement and transformation, according to a basic principle of science:

- *Necessity is the theme and inventor of nature, the curb and the rule.*

Da Vinci recognized the two basic forces of fluid mechanics to be inertial and viscous forces, realized that water is incompressible and though it assumes an infinite number of shapes, its mass and volume is always conserved. Below we will return to the following deep insights expressed by da Vinci:

- *In order to give the true science of the movements of the birds in the air, it is necessary to first give the science of the winds.*

- *As much force is exerted by the object against the air as by the air against the object.*

- *The spiral or rotary movement of every liquid is so much swifter as it is nearer to the center of revolution. What we are here proposing is a fact worthy of admiration, since the circular movement of a wheel is so much slower as it is nearer to the center of the rotating object.*

- *I have found among the excessive and impossible delsusions of men, the search for continuous motion, which is called by some the perpetual wheel.*

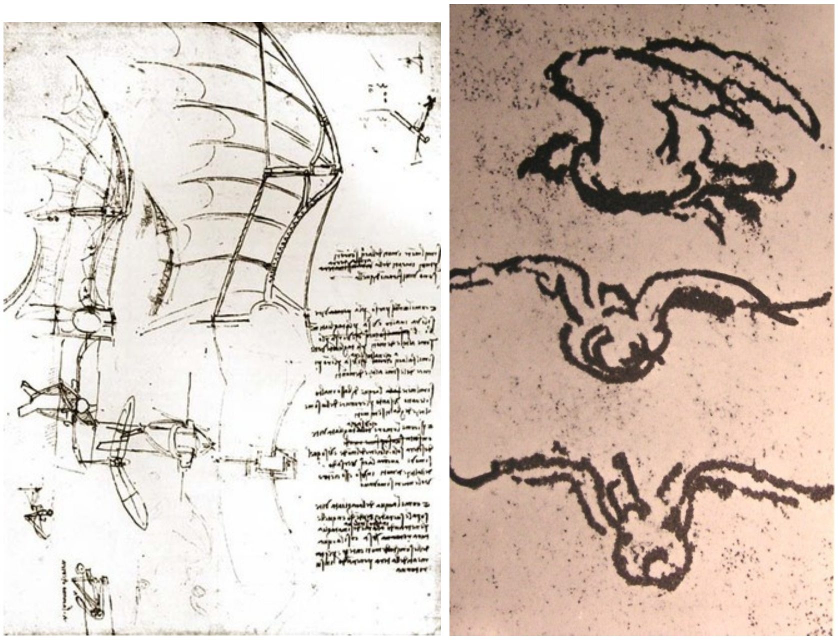


Figure 31.1: Da Vinci studies of bird wings and flight

31.4 The Mathematician

Da Vinci had a great admiration for mathematics:

- *A bird is an instrument working according to mathematical law, which is within the capacity of man to reproduce.*

- *There is no certainty, where one cannot apply any of the mathematical sciences, nor those which are connected with the mathematical sciences.*

- *Mechanics are the Paradise of mathematical science, because here we come to the fruits of mathematics.*
- *Let no man who is not a mathematician read my principles*.

Although da Vinci had little technical training in mathematics, he understood basic principles such as the law of free fall motion long before Galileo and conservation of mass:

- *The natural motion of heavy things, at each degree of its descent acquires a degree of velocity.*
- *If the water does not increase, nor diminish, in a river which may be of varying turtuoisities, breadths and dephts, the water will pass in equal quantities in equal times through every degree of the length of the river*

Da Vinci adopted Aristotle's principle of motion:

- *Of everything that moves, the space which it acquires is as great as that which it leaves*.

He also formulated basic priniciples of differential geometry:

- *The line is made with the movement of the point.*
- *The surface is made by the tramsversal movement of the line.*
- *The body is made by the movement of the extension of the surface.*

31.5 The Engineer

Between 1480 and 1505 da Vinci made a series of studies of birds and bats and developed sketches of flying machines, including gliders and more or less impossible devices including a flying machine like a boat. The pilot was intended to lie stretched out and to pull at oars which would propel the craft through air rather than water. Although this does not work for larger devices, this is essentially the mechanism for flight of small insects experiencing a substantial viscosity of air.

The modern helicopter invented by the Ukrainian-American engineer *Igor Sikorsky* in the 1930s, was probably inspired by a design by da Vinci with a helical screw instead of rotor blades (which of course did not work). The most inventive of da Vinci's flying machines was the glider in Fig.31.2 with the following control technique:

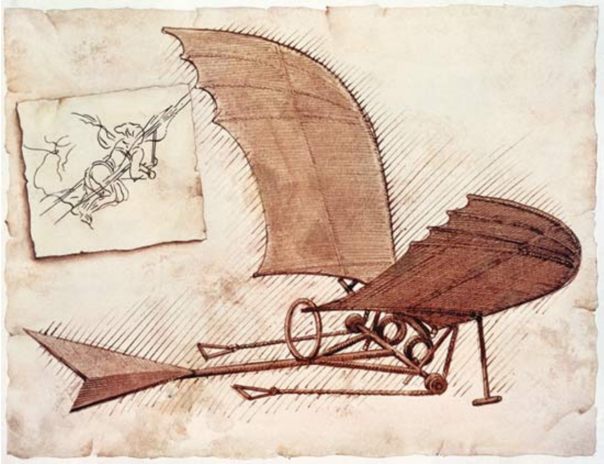


Figure 31.2: Da Vinci design of a glider

- *this [man] will move on the right side if he bends the right arm and extends the left arm; and he will then move from right to left by changing the position of the arms.*

31.6 The Philosopher

Da Vinci expressed a view on the interaction of body and soul connecting that of Descartes leading into modern conceptions of mind-brain interaction:

- *It could be said that such an instrument designed by man is lacking only the soul of the bird, which must be counterfeited with the soul of man...However, the soul of the bird will certainly respond better to the needs of its limbs than would the soul of the man, separated from them and especially from their almost imperceptible balancing movements*
- *Spiritual movement flowing through the limbs of sentient animals, broadens their muscles. Thus boadened, these muscles become shortened and draw back the tendons that are connected to them. This is the origin of force in human limbs...Material movement arises from the immaterial.*

Chapter 32

Newton's Incorrect Theory

> Dubito ergo cogito; cogito ergo sum. (Descartes)
>
> The foolish dog barks at the flying bird. (Bob Marley)
>
> The man who has no imagination has no wings. (Muhammad Ali)
>
> When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong. (Arthur C. Clarke)

We know that a surfing board or water skis can carry the weight of a person, but only in sufficiently rapid motion depending on the weight of the person and the area exposed to the water surface. The vertical force or lift is a reaction to a constant downward push of water as the board meets new water in its horsiontal motion. Without horisontal motion the board with the person will sink into the water. This is illustrated in Fig.**??**.

Newton was the first scientist to seek to develop a theory of lift and drag, and suggested that they should both be proportional to the density of the fluid and the square of the speed, which turns out to be more or less correct. Using a surfing board (or skipping stone) argument, which according to NASA we now know is wrong, Newton derived the above formula

$$L = \sin^2(\alpha)\rho U^2, \tag{32.1}$$

for the lift L of a tilted flat plate of unit area with a quadratic dependence on the angle of attack α. This formula follows from the fact that the mass $\rho U\sin(\alpha)$ hits the plate from below per unit time and gets redirected with a downward velocity

$U\sin(\alpha)$ corresponding to a change of momentum equal to $\sin^2(\alpha)\rho U^2$, which equals the lift force L by Newton's 2nd law.

Newton's formula explains the force acting on a surf board on water. The ratio of density of water to that of air is about 1000 and thus surfing on air requires about 30 times as large speed as surfing on water, because lift scales with the speed squared. Water skiing is possible at a speed of about 20 knots, which would require a speed about 1000 km/h, close to the speed of sound, for surfing on air. We understand that Newton's formula grossly under-estimates the lift, at least for subsonic speeds. We understand that flying *in* the air is not at all like surfing *on* water. Newton could thus prove that subsonic flight is impossible in theory, and so must have viewed the the flight of birds with surprise. Apparently birds were not willing to abide by the laws of Newtonian mechanics, but how could they take this liberty?

It is possible that Newton contributed to delaying human flight by making it seem impossible. Only after powered human flight had been demonstrated to be possible by the Wright brothers in 1903, did mathematicians replace Newton's erronous lift formula with a formula compatible with flight, although the derivation of the new formula again turned out to be incorrect, as we will discover below...

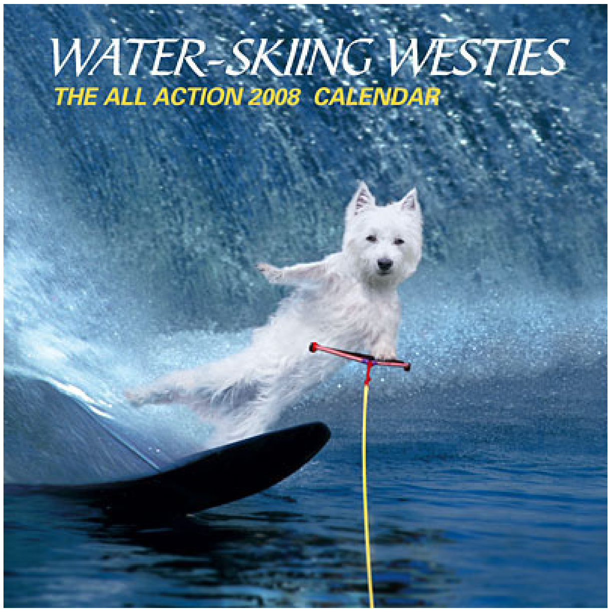


Figure 32.1: Incorrect explanation by Newton of lift by surfing.

Chapter 33

Robins and the Magnus Effect

> The greates part of military projectiles will at the time of their discharge acquire a whirling motion by rubbing against the inside of their respective pieces; and this whirling motion will cause them to strike the air very differently, from what the would do, had they no other but a progressive motion. By this means it will happen, that the resistance of the air will not always be directly opposed to their flight; but will frequently act in a line oblique to their course, and will thereby force them to deviate from the regular track, they would otherwise describe. (Robins in [65])

The English engineer Benjamin Robins (1707-1751), called the father of ballistics, introduced the concept of rifling the bore of guns to improve the accuracy of projectiles by spinning. In experiments with a whirling arm device he discovered that a spinning projectile experiences a transverse lift force, which he recorded in [65] in 1742. Euler translated Robins' book to German, but added a critical remark stating that on mathematical symmetry grounds the lift must be zero, and thus the measured lift must have been an effect of a non-symmetric projectile resulting from manufacturing irregularities. Recognized as the dominant hydrodynamicist of the eighteenth century, Euler far overshadowed Robins, and thus Robins' finding was not taken seriously for another century. In 1853 Gustav Magnus (1802-1870) in [62] suggested that the lift of a spinning ball, the so-called *Magnus effect*, was real and resulted from a whirlpool of rotating air around a ball creating a non-symmetric flow pattern with lift, an idea which was later taken up by Kutta and Zhukovsky as the decisive feature of their lift theory based on circulation.

Magnus suggested that, because of the whirlpool, for a topspin ball the air velocity would be larger below than above and thus result in a downward lift force because the pressure would be smaller below than above by Bernouilli. Magnus thus claimed to explain why a topspin tennis ball curves down, and a backspin curves up.

We shall see below that this explanation is incorrect: There is no whirlpool of rotating air around a spinning ball, nor is it any circulation around a wing. In both cases the lift has a different origin. We shall see that finding the real cause of the Magnus effect will lead us to an explanation of also the lift of a wing.

In 1749 Robins left the center stage of England, when he was appointed the engineer-general of the East India Company to improve the fortifications at St. David, Madras, where he died of fever at an early age of forty-four.

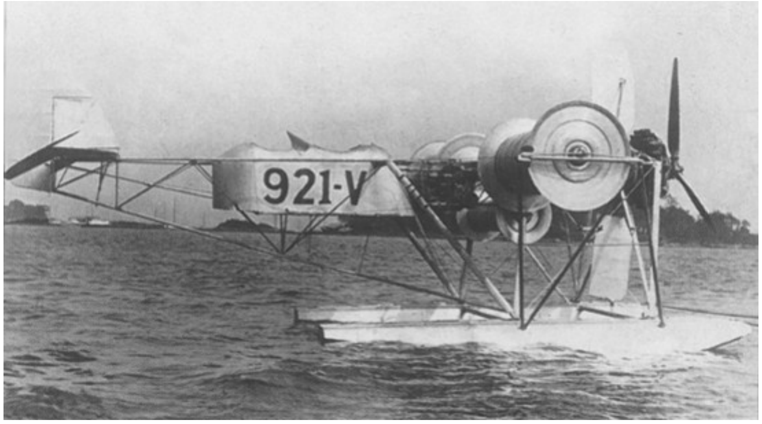


Figure 33.1: Built in 1930 (USA), the 921-V is reported to have been flown at least once - ending it's short carreer with a crash landing. Three cylinders with disks performing as winglets driven by a separate engine. Probably the only aircraft equipped with cylinder wings which made it into the air...

33.1 Early Pioneers

The flight of birds has always challenged human curiosity with the dream of human flight described already in the Greek myth about the inventor and master craftsman *Deadalus*, who built wings for himself and his son *Icarus* to escape from imprisonment in the Labyrinth of Knossos on the island of Crete. The leading scholar *Abbas Ibn Firnas* of the Islamic culture in Cordoba in Spain studied the mechanics of flight and in 875 AD survived one successful flight on a pair of wings made of feathers on a wooden frame. Some hundred years later the great Turkish scholar *Al-Djawhari* tied two pieces of wood to his arms and climbed the roof of a tall mosque in Nisabur, Arabia, and announced to a large crowd:

- *O People! No one has made this discovery before. Now I will fly before your very eyes. The most important thing on Earth is to fly to the skies. That I will do now.*

Unfortunately, he did not, but fell straight to the ground and was killed. It would take 900 years before the dream of Al-Djawhari became true, after many unsuccessful attempts. One of the more succesful was made by *Hezarfen Ahmet Celebi*, who in 1638 inspired by work of Leonardo da Vinci, after nine experimental attempts and careful studies of eagles in flight, took off from the 183 foot tall Galata Tower near the Bosphorus in Istanbul and successfully landed on the other side of the Bosphorus. The word Hezarfen means expert in 1000 sciences and a reward of 1000 gold pieces was given to Hezarfen for his achievement.

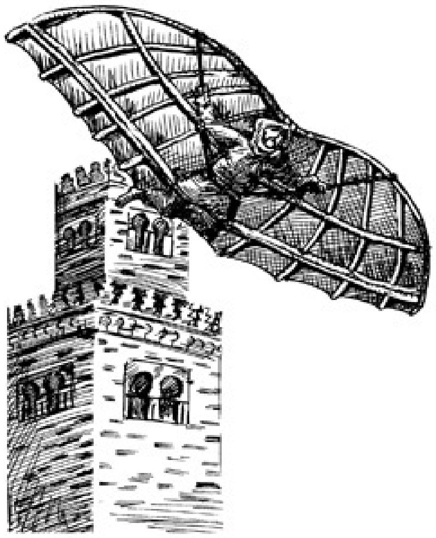


Figure 33.2: Abbas Ibn Firnas flying from the Mosque of Cordoba in 875 AD.

The understanding of *why* it is possible to fly has haunted scientists since the birth of mathematical mechanics in the 17th century. To fly, an upward force on the wing, referred to as *lift* L, has to be generated from the flow of air around the wing, while the air resistance to motion or *drag* D, is not too big. The mystery is *how* a sufficiently large ratio $\frac{L}{D}$ can be created. In the *gliding flight* of birds and airplanes with fixed wings at subsonic speeds, $\frac{L}{D}$ is typically between 10 and 20, which means that a good glider can glide up to 20 meters upon loosing 1 meter in altitude, or that *Charles Lindberg* could cross the Atlantic in 1927 at a speed of 50 m/s in his 2000 kg *Spirit of St Louis* at an effective engine thrust of 150 kp (with $\frac{L}{D} = 2000/150 \approx 13$) from 100 horse powers (because 1 hp = 75 kpm/s).

By elementary Newtonian mechanics, lift must be accompanied by *downwash* with the wing redirecting air downwards. The enigma of flight is the mechanism generating substantial downwash under small drag, which is also the enigma of sailing against the wind with both sail and keel acting like wings creating lift.

Classical mathematical mechanics could not give an answer. *Newton* computed the lift of a tilted flat plate (of unit area) redirecting a horisontal stream of fluid particles of speed U and density ρ, but obtained a disappointingly small value approximately proportional to the square of the tilting angle or *angle of attack* α (in radians with one radian $= \frac{\pi}{180}$ degrees):

$$L = \sin^2(\alpha)\rho U^2, \tag{33.1}$$

since $\sin(\alpha) \approx \alpha$ (for small α). The French mathematician *Jean le Rond d'Alembert* (1717-1783) followed up in 1752 with a computation based on *potential flow* (inviscid incompressible irrotational stationary flow), showing that both the drag and lift of a body of any shape (in particular a wing) is zero, referred to as *d'Alembert's paradox*, since it contradicts observations and thus belongs to fiction. To explain flight d'Alembert's paradox had to be resolved.

But the dream was not given up and experiments could not be stopped only because a convincing theory was lacking; undeniably it was possible for birds to fly without any theory, so maybe it could somehow be possible for humans as well.

The first published paper on aviation was *Sketch of a Machine for Flying in the Air* by the Swedish polymath *Emanuel Swedenborg*, published in 1716, describing a flying machine consisting of a light frame covered with strong canvas and provided with two large oars or wings moving on a horizontal axis, arranged so that the upstroke met with no resistance while the downstroke provided lifting power. Swedenborg understood that the machine would not fly, but suggested it as a start and was confident that the problem would be solved:

- *It seems easier to talk of such a machine than to put it into actuality, for it*

> *requires greater force and less weight than exists in a human body. The science of mechanics might perhaps suggest a means, namely, a strong spiral spring. If these advantages and requisites are observed, perhaps in time to come some one might know how better to utilize our sketch and cause some addition to be made so as to accomplish that which we can only suggest. Yet there are sufficient proofs and examples from nature that such flights can take place without danger, although when the first trials are made you may have to pay for the experience, and not mind an arm or leg.*

Swedenborg would prove prescient in his observation that powering the aircraft through the air was the crux of flying.

33.2 Cayley

The British engineer *Sir George Cayley* (1773-1857), known as *the father of aerodynamics*, was the first person to identify the lift and drag forces of flight, discovered that a curved lifting surface would generate more lift than a flat surface of equal area. and designed different gliders as shown in Fig.33.3. In 1804 Cayley designed and built a model monoplane glider of strikingly modern appearance. with a cruciform tail, a kite-shaped wing mounted at a high angle of incidence and a moveable weight to alter the center of gravity.

In 1810 Cayley published his now-classic three-part treatise *On Aerial Navigation*, the first to state that lift, propulsion and control were the three requisite elements to successful flight, and that the propulsion system should generate thrust while the wings should be shaped so as to create lift. Cayley observed that birds soared long distances by simply twisting their arched wing surfaces and deduced that fixed-wing machines would fly if the wings were cambered. Thus, one hundred years before the Wright brothers flew their glider, Cayley had established the basic principles and configuration of the modern airplane, complete with fixed wings, fuselage, and a tail unit with elevators and rudder, and had constructed a series of models to demonstrate his ideas. In 1849 Cayley built a large gliding machine, along the lines of his 1799 design, and tested the device with a 10-year old boy aboard. The gliding machine carried the boy aloft on at least one short flight.

Cayley recognized and searched for solutions to the basic problems of flight including the ratio of lift to wing area, determination of the center of wing pressure, the importance of streamlined shapes, the recognition that a tail assembly was essential to stability and control, the concept of a braced biplane structure for strength, the concept of a wheeled undercarriage, and the need for a lightweight source of power. Cayley correctly predicted that sustained flight would not occur until a lightweight engine was developed to provide adequate thrust and lift.

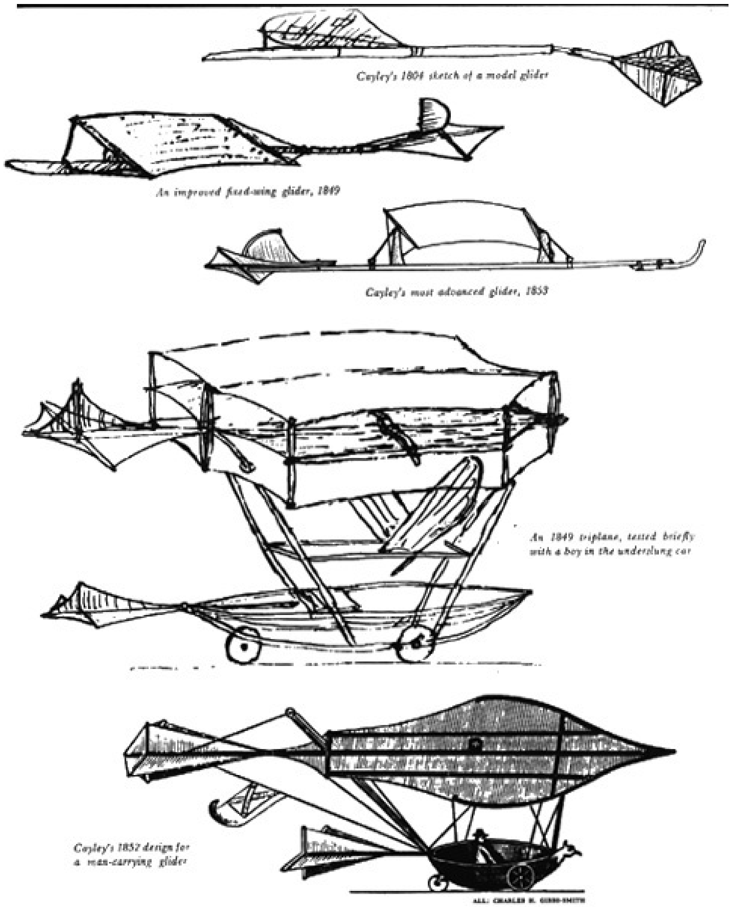

Cayley's 1804 sketch of a model glider
An improved fixed-wing glider, 1849
Cayley's most advanced glider, 1853
An 1849 triplane, tested briefly with a boy in the underslung car
Cayley's 1852 design for a man-carrying glider
ALL: CHARLES H. GIBBS-SMITH

Figure 33.3: Different gliders designed by Cayley.

Cayley's efforts were continued by *William Henson* who designed a large passenger-carrying steam-powered monoplane, with a wing span of 150 feet, named *The Henson Aerial Steam Carriage* for which he received a patent in 1843, but it could not fly. The Aerial Transit Company's publicist, Frederick Marriott, had a number of prints made in 1843 depicting the Aerial Steam Carriage over the pyramids in Egypt, in India, over London, England, and other places, which drew considerable interest from the press.

In 1856, the French aviator *Jean-Marie Le Bris* made the first flight higher than his point of departure, by having his glider *L'Albatros artificiel* pulled by a horse on a beach. He reportedly achieved a height of 100 meters, over a distance of 200 meters. In 1874, *Félix du Temple* built the *Monoplane*, a large plane made of aluminium in Brest, France, with a wingspan of 13 meters and a weight of only 80 kilograms (without pilot). Several trials were made with the plane, and it is generally recognized that it achieved lift off under its own power after a ski-jump run, glided for a short time and returned safely to the ground, making it the first successful powered flight in history, although the flight was only a short distance and a short time.

The British marine engineer *Francis Wenham* (1824-1908) discovered, while unsuccessfully attempting to build a series of unmanned gliders, that the most of the lift from a bird-like wing was generated at the leading edge, and concluded that long, thin wings would be better than the bat-like ones suggested by many, because they would have more leading edge for their weight. He presented a paper on his work to the newly formed Royal Aeronautical Society of Great Britain in 1866, and decided to prove it by building the world's first wind tunnel in 1871. Members of the Society used the tunnel and learned that cambered wings generated considerably more lift than expected by Cayley's Newtonian reasoning, with lift-to-drag ratios of about 5:1 at 15 degrees. This clearly demonstrated the ability to build practical heavier-than-air flying machines; what remained was the problem of controlling the flight and powering them.

In 1866 the Polish illiterate peasant *Jan Wnek* built and flew a controllable glider launching himself from a special ramp on top of the Odporyszow church tower 95 m high above the valley below, especially during religious festivals, carnivals and New Year celebrations.

33.3 Lilienthal and Wright

The German engineer *Otto Lilienthal* (1848-1896) expanded Wenham's work, made careful studies of the gliding flight of birds recorded in *Birdflight as the Basis of Aviation* [25] and designed a series of ever-better hang gliders allowing him to make 2000 successful heavier-than-air gliding flights starting from a little artificial hill, before in 1896 he broke his neck falling to the ground after having stalled at 15 meters altitude. Lilienthal rigorously documented his work, including photographs, and for this reason is one of the best known of the early pioneers.

The first sustained powered heavier-than-air flights were performed by the two

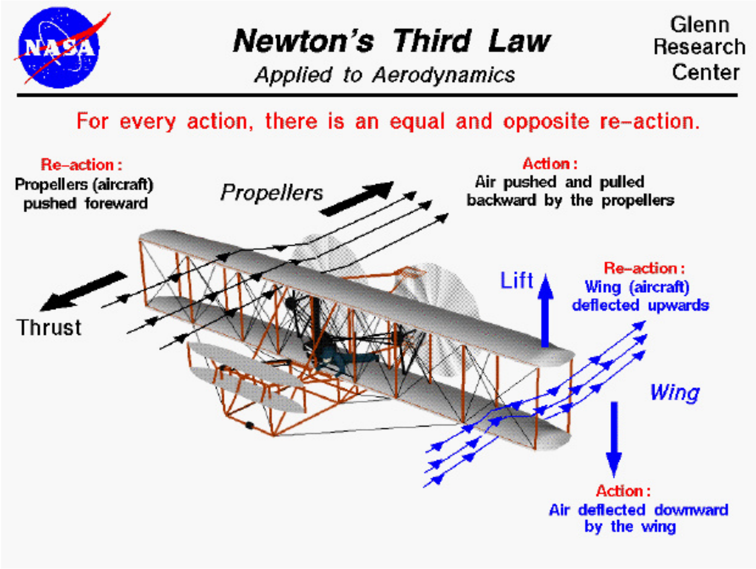

Newton's Third Law
Applied to Aerodynamics
Glenn Research Center
For every action, there is an equal and opposite re-action.
Re-action: Propellers (aircraft) pushed foreward
Propellers
Action: Air pushed and pulled backward by the propellers
Thrust
Lift
Re-action: Wing (aircraft) deflected upwards
Wing
Action: Air deflected downward by the wing

Figure 33.4: Tautological explanation of the flight of *The Flyer* by NASA.

brothers *Orwille* and *Wilbur Wright*, who on the windy dunes of Kill Devils Hills at Kitty Hawk, North Carolina, on December 17 in 1903, managed to get their 400 kg airplane *Flyer* off ground into sustained flight using a 12 horse power engine. The modern era of aviation had started.

33.4 Status Quo

It is natural to expect that today the mechanics of gliding flight is well understood. However, in the presentation [117] directed to math and science teachers, the authority NASA first dismisses the following three popular theories for lift of a wing as being incorrect:

- **longer path above**: low pressure above the wing because of higher velocity because of equal transit time above and below,
- **Venturi nozzle**: higher velocity and lower pressure above because of wing asymmetry,
- **skipping stone**: force from fluid particles hitting the wing from below,

with the first two being based on *Bernouilli's principle* combining low pressure with high velocity and vice versa, and the third on Newton's 3rd law. NASA then hints at a trivial tautological fourth theory similar to skipping stone, see the end of this chapter for details:

- **lift from flow turning**,

and ends with a disappointing seemingly out of reach:

- *To truly understand the details of the generation of lift, one has to have a good working knowledge of the Euler Equations.*

The Plane&Pilot Magazine [118] has the same message. The ambitious teacher thus does not get much help from expertize to explain the mystery of flight to curious students, not because NASA wants to hide the secret, but simply because there is no convincing scientific explanation of why it possible to fly in the existing literature, however incredible this may seem.

But how can this be? Can you design airplanes without understanding what makes it possible to fly? Yes, you can. Just look at the birds, who are able to fly without scientific understanding. You can replace theory by trial and error practice. This is what Lilienthal and the Wright brothers did. On the other hand, nothing is more practical than a good theory, but such a theory is missing for flight. In short, state-of-the-art literature [2, 98, 137, 120, 126] presents a

- **theory for drag without lift in viscous laminar flow**

by *Ludwig Prandtl* [119], called *the father of modern fluid mechanics* [138], and a

- **theory for lift without drag in inviscid potential flow**

at small angles of attack by the mathematicians *Martin Kutta* and *Nikolai Zhukovsky*, called *the father of Russian aviation*. Kutta and Zhukovsky augumented inviscid zero-lift potential flow by a large scale *circulation* of air around the wing section causing the velocity to increase above and decrease below the wing as illustrated in Fig.33.6, thus generating lift proportional to the angle of attack [137, 126, 47]:

$$L = 2\pi\alpha\rho U^2. \tag{33.2}$$

Kutta-Zhukovsky thus showed that if there is circulation then there is lift, which by a scientific community in desperate search for a theory of lift after the flights by the Wright brothers in 1903, was interpreted as an equivalence ([126], p.94):

- *If the airfoil experiences lift, a circulation must exist.*

State-of-the-art is described in [2] as:

- *The circulation theory of lift is still alive... still evolving today, 90 years after its introduction.*

However, there is no

- **theory for lift and drag in turbulent incompressible flow**

such as the flow of air around a wing of a jumbojet at the critical phase of take-off at large angle of attack (12 degrees) and subsonic speed (270 km/hour). The first start of the new jumbojet Airbus 380 must have been a thrilling experience for the design engineers, since full scale tests cannot be made.

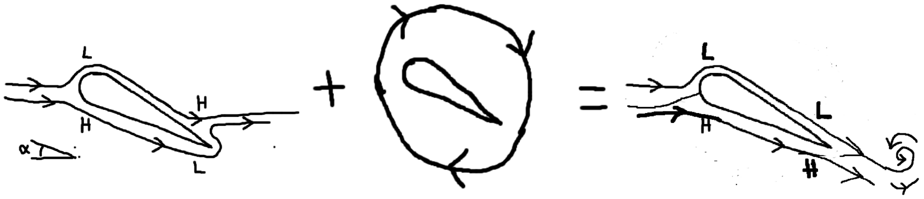


Figure 33.5: High (H) and low (L) pressure distributions of potential flow (left) past a wing section with zero lift/drag modified by circulation around the section (middle) to give Kutta-Zhukovsky flow (right) leaving the trailing edge smoothly with downwash/lift and a so-called starting vortex behind.

We observe that the real turbulent flow like potential flow adheres to the upper surface beyond the crest and thereby gets redirected, because the real flow is close to potential before separation, and potential flow can only separate at a point of stagnation with opposing flows meeting in the rear, as shown in [104, 102]. On the other hand, a flow with a viscous no-slip boundary layer will (correctly according to Prandtl) separate on the crest, because in a viscous boundary layer the pressure gradient normal to the boundary vanishes and thus cannot contribute the normal acceleration required to keep fluid particles following the curvature of the boundary after the crest [134]. It is thus the slip boundary condition modeling a turbulent boundary layer in slightly viscous flow, which forces the flow to suck to the upper surface and create downwash, as analyzed in detail in [134], and not any Coanda effect [47].

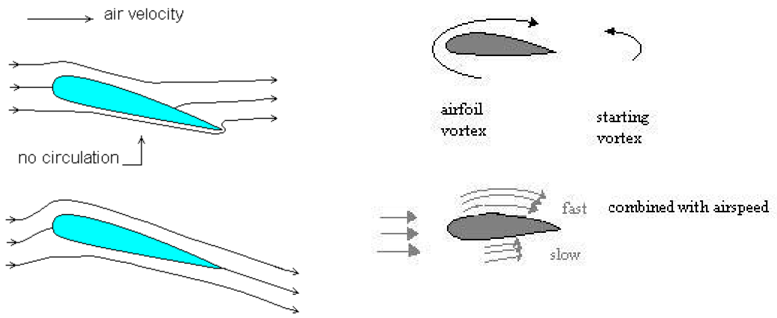


Figure 33.6: Lift theory by circulation as typically presented at typical web sources such as 1 and and 2.

This explains why gliding flight is possible for airplanes and larger birds, because the boundary layer is turbulent and acts like slip preventing early separation, but not for insects because the boundary layer is laminar and acts like no-slip allowing early separation. The *Reynolds number* of a jumbojet at take-off is about 10^8 with turbulent skin friction coefficient < 0.005 contributing less than 5% to drag, while for an insect with a Reynolds number of 10^2 viscous laminar effects dominate.

Chapter 34

Lilienthal and Bird Flight

> To invent an airplane is nothing. To build one is something. But to fly is everything. (Lilienthal)

> Sacrifices must be made! (Lilienthal near death after breaking his spine in an airplane crash in a glider of his design).

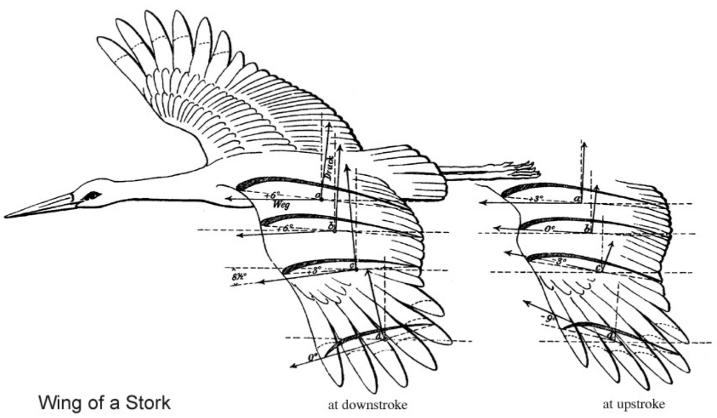
> No one can realize how substantial the air is, until he feels its supporting power beneath him. It inspires confidence at once. (Lilienthal)

> We returned home, after these experiments, with the conviction that sailing flight was not the exclusive prerogative of birds. (Lilienthal, 1874)

Otto Lilienthal gives in *Bird Flight as a Basis of Aviation* [25] the following description of bird flight:

- *From all the foregoing results it appears obvious that in order to discover the principles which facilitate flight, and to eventually enable man to fly, we must take the bird for our model. A specially suitable species of birds to act as our model is the sea-gull.*

- *How does the gull fly? At the very first glance we notice that the slender, slightly curved wings execute a peculiar motion, in so far as only the wing-tips move appreciably up and down, whilst the broader arm-portions near the body take little part in this movement, a condition of things which is illustrated in Fig. 76.*



Otto Lilienthal: Der Vogelflug als Grundlage der Fliegekunst,
Publishing house R. Gärtner, Berlin 1889, reprinting, publishing company Oldenbourg Munich 1943

Figure 34.1: Lilienthal's analysis of a stork in flight.

- *May we not assume that the comparatively motionless parts of the wings enable the gull to sail along, whilst the tips, consisting of easily rotating feathers, serve to compensate for the loss of forward velocity ? It is unmistakable that the wide Portion of the wing close to the body, which does little work and has little movement, is intended for sustaining, whilst the narrower tips, with their much greater amplitude of movement, have to furnish the tractive power necessary to compensate for the resistance of the bird's body and for any possible restraining component.*

- *This being conceded, we are forced to consider the flying apparatus of the bird as a most ingenious and perfect mechanism, which has its fulcrum in the shoulder joint, which moves up and down, and by virtue of its articulation permits of increased lift or fall as well as of rotation of the light tips.*

- *The arm portion of the wing is heavy, containing bones, muscles , and tendons, and therefore opposes considerable inertia to any rapid movement. But it is well fitted for supporting, because being close to the body, the air pressure upon it acts on a short lever arm, and the bending strain is therefore less severe on the wing. The tip is very light, consisting of feathers only, and can be lifted and depressed in rapid succession. If the air pressure produced by it increased in proportion to the greater amplitude of movement,* *it would require a large amount of work; and would also unduly strain the wings; we therefore conclude that the real function of the wing-tips is not so much the generation of a great lifting effect, but rather the production of a smaller, but tractive effect directed forward.*

- *In fact, actual observation leaves no doubt on this point. It is only necessary to watch the gull during sunshine, and from the light effects we tan distinctly perceive the changing inclination of the wing-tips, as shown in Figs. 77 and 78, which refer to the upstroke and downstroke of the wings respectively . The gull , flying away from us, presents at the upstroke, Fig. 77, the upper side of its wings strongly illuminated by the sun, whilst during the downstroke (Fig. 78) we have tlie shaded camber presented to us from the back. The tip evidently ascends with the leading edge raised, and descends with the leading edge depressed, both phases resulting in a tractive effect.*

Da Vinci had made similar observations in *Codex on Bird Flight*:

- *Those feathers which are farthest from their fastening will be the most flexible; then the tops of the feathers of the wings will be always higher than their origins, so that we may with reason say, that the bones of the wings will be lower in the lowering of the wings than any other part of the wings, and in the raising these bones of the wing will always be higher than any other part of such a wing. Because the heaviest part always makes itself the guide of the movement.*

- *The kite and other birds which beat their wings little, go seeking the course of the wind, and when the wind prevails on high then they will be seen at a great height, and if it prevails low they will hold themselves low.*

- *When the wind does not prevail in the air, then the kite beats its wings several times in its flight in such a way that it raises itself high and acquires a start, with which start, descending afterwards a little, it goes a long way without beating its wings, and when it is descended it does the same thing over again, and so it does successively, and this descent without flapping the wings serves it as a means of resting itself in the air after the aforesaid beating of the wings.*

- *When a bird which is in equilibrium throws the centre of resistance of the wings behind the centre of gravity, then such a bird will descend with its head down. This bird which finds itself in equilibrium shall have the centre* *of resistance of the wings more forward than the bird's centre of gravity, then such a bird will fall with its tail turned to the earth.*

- *When the bird is in the position and wishes to rise it will raise its shoulders and the air will press between its sides and the point of the wings so that it will be condensed and will give the bird the movement toward the ascent and will produce a momentum in the air, which momentum of the air will by its condensation push the bird up.*

The observations of the wingbeat cycle by da Vinci and Lilienthal da Vinci can be summarized as follows:

- a forward downstroke with the wing increasingly twisted towards the tip with the leading edge down,
- a backward upstroke with the wing twisted the other way with the leading edge up.

Below we shall analyze the lift and drag generated at different moments of the wingbeat cycle, and thus give a scientific explanation of the secret of bird flight. We compare with with the lack of a scientific theory of bird flight according to state-of-the-art [17]:

- *Always there have been several different versions of the flapping flight theory. They all exist in parallel and their specifications are widely distributed. Calculating the balance of forces even of a straight and merely slowly flapping wing remained difficult to the present day. In general, it is only possible in a simplified way. Furthermore, the known drives mechanism and especially wing designs leave a lot to be desired. In every respect ornithopters are still standing at the beginning of their design development.*

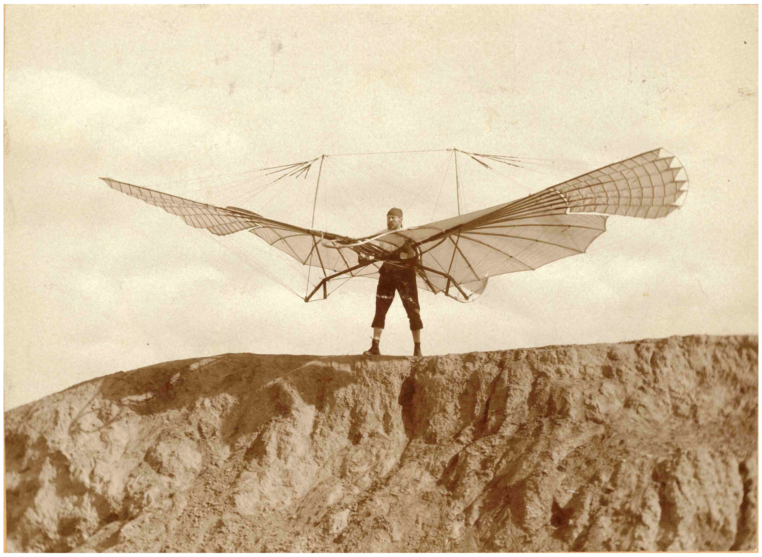


Figure 34.2: Lilienthal getting ready to simulate a stork in flight.

Chapter 35

Wilbur and Orwille Wright

> It is possible to fly without motors, but not without knowledge and skill. (Wilbur Wright)
>
> The desire to fly is an idea handed down to us by our ancestors who...looked enviously on the birds soaring freely through space...on the infinite highway of the air. (Wilbur Wright)
>
> The natural function of the wing is to soar upwards and carry that which is heavy up to the place where dwells the race of gods. More than any other thing that pertains to the body it partakes of the nature of the divine. (Plato in *Phaedrus*)
>
> Sometimes, flying feels too godlike to be attained by man. Sometimes, the world from above seems too beautiful, too wonderful, too distant for human eyes to see . . . (Charles Lindbergh in *The Spirit of St. Louis*)
>
> More than anything else the sensation is one of perfect peace mingled with an excitement that strains every nerve to the utmost, if you can conceive of such a combination. (Wilbur Wright)
>
> The exhilaration of flying is too keen, the pleasure too great, for it to be neglected as a sport. (Orwille Wright)

The first successful powered piloted controled flight was performed by the brothers Orwille and Wilbur Wright on December 17 1903 on the windy fields of Kitty Hawk, North Carolina, with Orwille winning the bet to be the pilot of the *Flyer* and Wilbur watching on ground, see Fig 35.1. In the words of the Wright brothers from Century Magazine, September 1908:

- *The flight lasted only twelve seconds, a flight very modest compared with that of birds, but it was, nevertheless, the first in the history of the world in which a machine carrying a man had raised itself by its own power into the air in free flight, had sailed forward on a level course without reduction of speed, and had finally landed without being wrecked. The second and third flights were a little longer, and the fourth lasted fifty-nine seconds, covering a distance of 852 feet over the ground against a twenty-mile wind.*

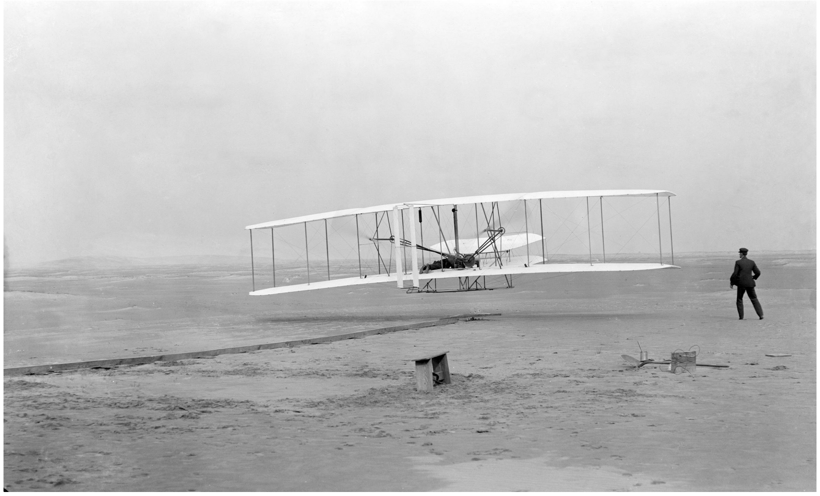


Figure 35.1: Orwille Wright (1871-1948) and Wilbur Wright (1867-1912) and the lift-off at Kitty Hawk, North Carolina, the 17th December 1903.

The work preceeding the success was described by Wilbur Wright in an address to the Western Society of Engineers in 1901 entitled *Some Aeronautical Experiments*:

- *The difficulties which obstruct the pathway to success in flying-machine construction are of three general classes: (1) Those which relate to the construction of the sustaining wings; (2) those which relate to the generation and application of the power required to drive the machine through the air; (3) those relating to the balancing and steering of the machine after it is actually in flight. Of these difficulties two are already to a certain extent solved. Men already know how to construct wings or aeroplanes which, when driven through the air at sufficient speed, will not only sustain the weight of the wings themselves, but also that of the engine and of the engineer as well. Men also know how to build engines and screws of sufficient* *lightness and power to drive these planes at sustaining speed. As long ago as 1884 a machine weighing 8,000 pounds demonstrated its power both to lift itself from the ground and to maintain a speed of from 30 to 40 miles per hour, but failed of success owing to the inability to balance and steer it properly. This inability to balance and steer still confronts students of the flying problem, although nearly eight years have passed. When this one feature has been worked out, the age of flying machines will have arrived, for all other difficulties are of minor importance.*

- *The person who merely watches the flight of a bird gathers the impression that the bird has nothing to think of but the flapping of its wings. As a matter of fact this is a very small part of its mental labor. To even mention all the things the bird must constantly keep in mind in order to fly securely through the air would take a considerable part of the evening. If I take this piece of paper, and after placing it parallel with the ground, quickly let it fall, it will not settle steadily down as a staid, sensible piece of paper ought to do, but it insists on contravening every recognized rule of decorum, turning over and darting hither and thither in the most erratic manner, much after the style of an untrained horse. Yet this is the style of steed that men must learn to manage before flying can become an everyday sport. The bird has learned this art of equilibrium, and learned it so thoroughly that its skill is not apparent to our sight. We only learn to appreciate it when we try to imitate it. Now, there are two ways of learning to ride a fractious horse: One is to get on him and learn by actual practice how each motion and trick may be best met; the other is to sit on a fence and watch the beast a while, and then retire to the house and at leisure figure out the best way of overcoming his jumps and kicks. The latter system is the safest, but the former, on the whole, turns out the larger proportion of good riders. It is very much the same in learning to ride a flying machine; if you are looking for perfect safety, you will do well to sit on a fence and watch the birds; but if you really wish to learn, you must mount a machine and become acquainted with its tricks by actual trial.*

- *Herr Otto Lilienthal seems to have been the first man who really comprehended that balancing was the first instead of the last of the great problems in connection with human flight. He began where others left off, and thus saved the many thousands of dollars that it had theretofore been customary to spend in building and fitting expensive engines to machines which were*

> *uncontrollable when tried. He built a pair of wings of a size suitable to sustain his own weight, and made use of gravity as his motor. This motor not only cost him nothing to begin with, but it required no expensive fuel while in operation, and never had to be sent to the shop for repairs. It had one serious drawback, however, in that it always insisted on fixing the conditions under which it would work. These were, that the man should first betake himself and machine to the top of a hill and fly with a downward as well as a forward motion. Unless these conditions were complied with, gravity served no better than a balky horse – it would not work at all. Although Lilienthal must have thought the conditions were rather hard, he nevertheless accepted them till something better should turn up; and in this manner he made some two thousand flights, in a few cases landing at a point more than 1,000 feet distant from his place of starting. Other men, no doubt, long before had thought of trying such a plan. Lilienthal not only thought, but acted; and in so doing probably made the greatest contribution to the solution of the flying problem that has ever been made by any one man. He demonstrated the feasibility of actual practice in the air, without which success is impossible. Herr Lilienthal was followed by Mr. Pilcher, a young English engineer, and by Mr. Chanute, a distinguished member of the society I now address. A few others have built gliding machines, but nearly all that is of real value is due to the experiments conducted under the direction of the three men just mentioned.*

The Wrights built The Flyer in 1903 using spruce and ash covered with muslin, with wings designed with a 1-in-20 camber. Since they could not find a suitable automobile engine for the task, they commissioned their employee Charlie Taylor to build a new design from scratch. A sprocket chain drive, borrowing from bicycle technology, powered the twin propellers, which were also made by hand. The Flyer was a canard biplane configuration. As with the gliders, the pilot flew lying on his stomach on the lower wing with his head toward the front of the craft in an effort to reduce drag, and steered by moving a cradle attached to his hips. The cradle pulled wires which warped the wings and turned the rudder simultaneously for lateral control, while a forward horisontal stabilizer (forward canard) was controled by the left hand.

To sum up, the Wright brothers were the first to solve the combined problem of (1) generation of lift by (sufficiently large) wings, (2) generation of thrust by a propeller powerd by a (sufficiently light) combustion engine and (3) horisontal control of balance under different speeds and angles of attack as well as lateral control. The data of the Flyer were:

- wingspan: 12.3 m
- wing area: 47 m^2
- length: 6.4 m
- height: 2.8 m
- weight (empty): 274 kg
- engine: gasoline 12 hp

At a lift/drag ratio of 10 the drag would be about 35 kp to carry a total weight of 350 kp, which at a speed of 10 m/s would require about 5 effective hp.

The Flyer had a forward canard for horisontal control, like the modern Swedish jet fighter *JAS Gripen*, which is an unstable configuration requiring careful control to fly, but allowing quick turns. The Wrights later replaced the canard with the conventional aft tail to improve stability. The stability of an airplane is similar to that of a boat, with the important design feature being the relative position of the center of gravity and the center of the forces from the fluid (center of buoyancy for a boat), with the center of gravity ahead (below) giving stability.

Chapter 36

Lift by Circulation

> Man must rise above the Earthto the top of the atmosphere and beyondfor only thus will he fully understand the world in which he lives. (Socrates)
>
> A single lifetime, even though entirely devoted to the sky, would not be enough for the study of so vast a subject. A time will come when our descendants will be amazed that we did not know things that are so plain to them. (Seneca)
>
> All the perplexities, confusion and distress in America arise, not from defects in their Constitution or Confederation, not from want of honor or virtue, so much as from the downright ignorance of the nature of coin, credit and circulation. (John Adams)
>
> If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things. (Descartes)

36.1 Lanchester

Frederick Lanchester, (1868-1946) was an English polymath and engineer who made important contributions to automotive engineering, aerodynamics and co-invented the field of operations research. He was also a pioneer British motor car builder, a hobby he eventually turned into a successful car company, and is considered one of the *big three* English car engineers, the others being Harry Ricardo and Henry Royce.

Lanchester began to study aeronautics seriously in 1892, eleven years before the first successful powered flight. Whilst crossing the Atlantic on a trip to the

United States, Lanchester studied the flight of herring gulls, seeing how they were able to use motionless wings to catch up-currents of air. He took measurements of various birds to see how the centre of gravity compared with the centre of support. As a result of his deliberations, Lanchester, eventually formulated his circulation theory, which still serves as the basis of modern lift theory. In 1894 he tested his theory on a number of models. In 1897 he presented a paper entitled *The soaring of birds and the possibilities of mechanical flight* to the Physical Society, but it was rejected, being too advanced for its time. Lanchester realised that powered flight required an engine with a far higher power to weight ratio than any existing engine. He proposed to design and build such an engine, but was advised that no one would take him seriously [1].

Stimulated by Lilienthal's successful flights and his widely spread book *Bird Flight as the Basis of Aviation* from 1899, the mathematician *Martin Kutta* (1867-1944) in his thesis presented in 1902 modified the erronous classical potential flow solution by including a new term corresponding to a rotating flow around the wing with the strength of the vortex determined so that the combined flow velocity became zero at the trailing edge of the wing. This *Kutta condition* reflected the observation of Lilienthal that the flow should come off the wing smoothly, at least for small angles of attack. The strength of the vortex was equal to the *circulation* around the wing of the velocity, which was also equal to the lift. Kutta could this way predict the lift of various wings with a precision of practical interest. But the calculation assumed the flow to be fully two-dimensional and the wings to be very long and became inaccurate for shorter wings and large angles of attack.

36.2 Kutta

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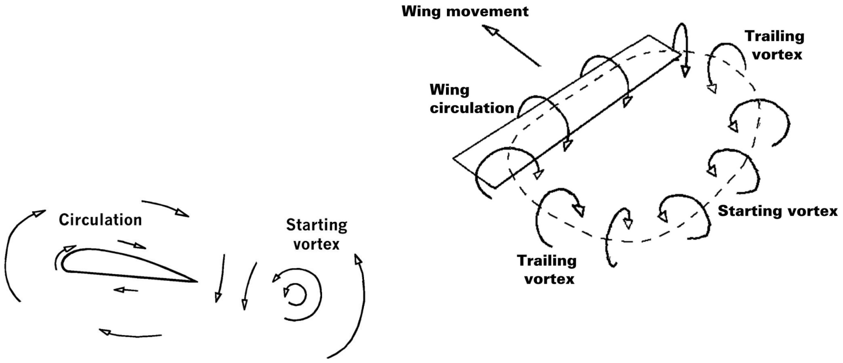

Figure 36.1: Generation of lift according to Kutta-Zhukovsky theory, as extended by Prandtl by connecting the circulation around the wing to the starting vortex by so-called trailing vortices from the wing tips. The circulation around the wing and starting vortices are unphysical, while the trailing vortices from the wing tips are real and often can be observed by condensation in damp weather.

long and became inaccurate for shorter wings and large angles of attack.

36.3 Zhukovsky

The mathematician *Nikolai Zhukovsky* (1847-1921), called the *father of Russian aviation*, in 1906 independently derived the same mathematics for computing lift as Kutta, after having observed several of Lilienthal's flights, which he presented before the Society of Friends of the Natural Sciences in Moscow as:

- *The most important invention of recent years in the area of aviation is the flying machine of the German engineer Otto Lilienthal.*

Zhukovsky also purchased one of the eight gliders which Lilienthal sold to members of the public.

Kutta and Zhukovsky thus could modify the mathemathical potential theory of lift of a wing to give reasonable results, but of course could not give anything but a very heuristic justification of their Kutta-Zhukovsky condition for the velocity at the trailing edge of the wing, and could not treat realistic wings in three dimensions. Further, their modified potential solutions are not turbulent, and as we will see below, their calculations were merely happy coincidences (knowing ahead the correct answer to obtain) without connection to the physics of real turbulent flow: There is no circulation around a wing, and connecting lift to circulation is unphysical.

It is remarkable that 400 years passed between Leonardo da Vinci's investigations and the largely similar ones by Lilienthal. Why did it take so long time from almost success to success? What was the role of the misleading mathematics of Newton and d'Alembert, still influencing the judgement of e.g. Lord Kelvin in the late 19th century?

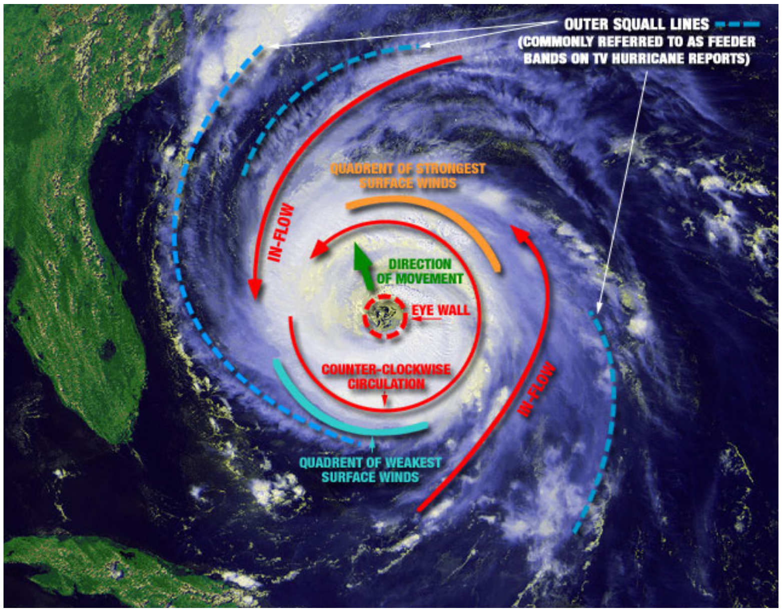
OUTER SQUALL LINES
(COMMONLY REFERRED TO AS FEEDER BANDS ON TV HURRICANE REPORTS)
QUADRENT OF STRONGEST SURFACE WINDS
IN-FLOW
DIRECTION OF MOVEMENT
EYE WALL
COUNTER-CLOCKWISE CIRCULATION
IN-FLOW
QUADRENT OF WEAKEST SURFACE WINDS

Figure 36.2: Hurricane with physical circulation.

Chapter 37

The Disastrous Legacy of Prandtl

37.1 Critique by Lancaster and Birkhoff

Prandtl's contribution to fluid mechanics was to explain separation, drag and lift as effects of a very small (vanishingly small) viscosity. This view has been seriously questioned, however with little effect since no alternative to Prandtl's theory has been in sight. Lancaster states already in 1907 in his in *Aerodynamics*[44]:

- *According to the mathematical theory of Euler and Lagrange, all bodies are of streamline form (with zero dragh and lift). This conclusion, which would otherwise constitute a reductio ad absurdum, is usually explained on the gorund the fluid of theory is inviscid, whereas real possess viscosity. It is questionable of this expanlanation alone is adequate.*

Birkhoff follows up in his *Hydromechanics* from 1950 [92]:

- *The art of knowing "how to apply" hydrodynamical theories can be learned even more effectively, in my opinion, by studying the paradoxes I will describe (e.g d'Alemberts paradox). Moreover, I think that to attribute them all to the neglect of viscosity is an oversimplification. The root lies deeper, in lack of precisely that deductive rigor whose importance is so commonly minimized by physicists and engineers.*

However, critique of Prandtl was not well received, as shown in the review of Birkhoff's book by James. J. Stoker [86]. The result is that Prandtl still dominates fluid mechanics today, although the belief in Prandtl's boundary layer theory (BLT) seems to be fading as expressed by Cowley [93]:

- *But is BLT a 20th century paradox? One may argue, yes, since for quantitative agreement with experiment BLT will be outgunned by computational fluid dynmaics in the 21st century.*

The 21st century is now here, and yes, computational fluid mechanics reveals a different scenario than Prandtl's.

But Prandtl's influence is still strong, as evidenced by the common belief that accurate computational simulation requires very thin boundary layers to be resolved. Thus Kim and Moin [43] claim that to correctly predict lift and drag of an aircraft at the relevant Reynolds number of size 10^8, requires computation on meshes with more than 10^{16} mesh points, which is way out of reach for any foreseeable computer. This puts CFD into a deadlock: Either compute at irrelevant too small Reynolds numbers or invent turbulence models, which has shown to be very difficult.

Techniques for preventing laminar separation based on suction and blowing have been suggested. In the recent study [128] computational simulations are presented of synthetic jet control for a NACA 0015 wing at Reynolds number 896.000 (based on the chord length) for different angles of attack. As indicated, the relevant Reynolds number is two orders of magnitude larger, and the relevance of the study can be questioned. The effects of the synthetic jet control may simply be overshadowed by turbulent boundary layers.

37.2 Can You Prove that Prandtl Was Incorrect?

Lancaster and Birkhoff did not accept Prandtl's explanation of the generation of drag and lift as an effect of a vanishingly thin boundary layer. We have said that it is difficult to directly prove that an infinitely small cause cannot have a large effect, without access to an infinitely precise mathematical model or laboratory, which are not available. So Prandtl can be pretty safe to direct attacks, but not to indirect: Suppose you eliminate that vanishingly small cause from the consideration altogether, and yet obtain good correspondence between theory and experiment, that is, suppose you observe the effect without the infinitely small cause. Then you can say that the small cause has little to do with the effect.

This is what we do: We compute turbulent solutions of the incompressible Navier-Stokes equations with *slip* boundary conditions, requiring only the normal velocity to vanish letting the tangential velocity be free, and we obtain drag and lift which fit with experiments. We thus obtain the effect (drag and lift) without

Prandtl's cause consisting of a viscous boundary layer with *no-slip* boundary condition requiring also the tangential velocity to vanish. We conclude that the origin of drag and lift in slightly viscous flow, is not viscous boundary layers with no-slip boundary conditions.

We have motivated the use of slip boundary condition by the fact that the *skin friction* of a turbulent boundary layer (the tangential force from a no-slip boundary condition), tends to zero with the viscosity, which is supported by both experiment and computation, also indicating that boundary layers in general are turbulent. More generally, we use a friction-force boundary condition as a model of the skin friction effect of a turbulent boundary layer, with a (small) friction coefficient determined by the *Reynolds number* $Re = \frac{UL}{\nu}$, where U is a representative velocity, L a length scale and ν the viscosity. The limit case of zero friction with slip then corresponds to vanishing viscosity/very large Reynolds number, while large friction models no-slip of relevance for small to moderately large Reynolds numbers. In mathematical terms we combine the Navier-Stokes equations with a natural (Neumann/Robin type) boundary condition for the tangential stress, instead of an essential (Dirichlet type) condition for the tangential velocity as Prandtl did.

Chapter 38

Prandtl and Boundary Layers

> On an increase of pressure, while the free fluid transforms part of its kinetic energy into potential energy, the transition layers instead, having lost a part of their kinetic energy (due to friction), have no longer a sufficient quantity to enable them to enter a field of higher pressure, and therefore turn aside from it. (Prandtl)
>
> The modern world of aerodynamics and fluid dy- namics is still dominated by Prandtls idea. By every right, his boundary-layer concept was worthy of the Nobel Prize. He never received it, however; some say the Nobel Committee was reluctant to award the prize for accomplish- ments in classical physics...(John D. Anderson in [2])
>
> No flying machine will ever fly from New York to Paris. (Orville Wright)

38.1 Separation

The generation of lift and drag of a wing is closely connected to problem of *separation* in fluid mechanics: As a body moves through a slightly viscous fluid initially at rest, like a car or airplane moving through still air, or equivalently as a fluid flows around a body at rest, fluid particles are deviated by the body in a *contracting flow* switching to an *expanding flow* at a *crest* and eventually *separate* away from the body somewhere in the rear, at or after the crest. In the front there is typically a *stagnation point*, where the fluid velocity vanishes allowing *laminar attachment at stagnation* to the boundary. On the other hand the fluid mechanics of the *turbulent separation* occuring in the rear in slightly viscous flow, which creates *drag* and *lift* forces, appears to be largely unknown, despite its crucial importance in many applications, including flying and sailing. The basic study concerns separation from a convex body like a sphere, circular cylinder, wing, car or boat hull.

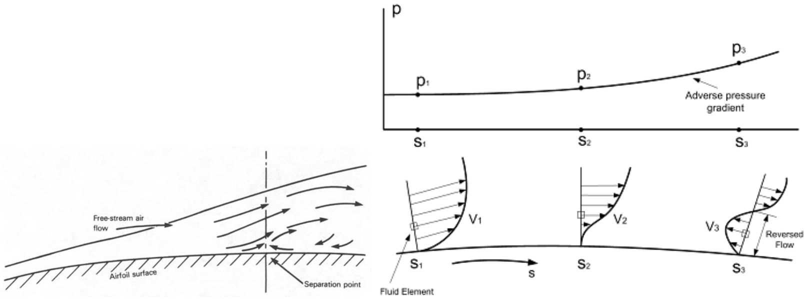

Figure 38.1: Prandtl's idea of laminar viscous separation with no-slip caused by an adverse pressure gradient, which is does not describe the turbulent slightly viscous separation with slip in the flow of air around a wing.

38.2 Boundary Layers

In 1904 the young German physicist Ludwig Prandtl (1875-1953) suggested in a 10 page sketchy presentation entitled *Motion of Fluids with Very Little Viscosity* [119] at the Third International Congress of Mathematics in Heidelberg, that the substantial drag of a bluff body moving through a fluid with very small viscosity (such as air or water), possibly could arise from the presence of of a thin *laminar boundary layer*, where the fluid velocity radpidly changes from its free-stream value to zero on the boundary corresponding to a *no-slip boundary condition*, causing the flow to *separate* from the boundary brought to stagnation under an *adverse pressure gradient* (negative pressure gradient in the flow direction), to form a low-pressure wake behind the body. But the acceptance of Prandtl's ideas was slow [2]:

- *Prandtls idea (about the boundary layer) went virtually unnoticed by anybody outside of Göttingen... The fifth and sixth editions of Lambs classic text Hydrodynamics published in 1924, devoted only one paragraph to the boundary-layer concept.*

However, Prandtl had two forceful students, Theodore von Karman (who emigrated to the US in 1930) and Hermann Schlichting (who stayed in Germany), who crowned Prandtl as the father of modern fluid mechanics. Prandtl's main ideas are described as follows in Schlichting's treatise *Boundary Layer Theory* from 1951:

- *Boundary layer flow has the peculiar property that under certain conditions the flow in the immediate neighbourhood of a solid wall becomes reversed causing the boundary layer to separate from it. This is accompanied by a more or less pronounced formation of eddies in the wake of the body. Thus the pressure distribution is changed and differs markedly from that in a frictionless stream. The deviation in pressure distribution from that of the ideal is the cause of form drag, and its calculation is thus made possible with the aid of boundary layer theory.*

- *The first important question to answer is to find when separation of the flow from the wall may occur. When a region with an adverse pressure gradient exists along the wall, the retarded fluid particles cannot, in general, penetrate too far into the region of increased pressure owing to their small kinetic energy. Thus the boundary layer is deflected sideways from the wall, separates from it, and moves into the main stream. In general the fluid particles follow the pressure gradient and move in a direction opposite to the external stream.*

- *In some cases the boundary layer increases its thickness considerably in the downstream direction and the flow in the boundary layer becomes reversed. This causes the decelerated fluid particles to be forced outwards, which means that the boundary layer is separated from the wall. We then speak of boundary layer separation. This phenomenon is always associated with the formation of vortices and with large energy losses in the wake of the body. The large drag can be explained by the existence of large deviation in pressure distribution (from potential flow), which is a consequence of boundary-layer separation.*

- *Downstream the pressure minimum the discrepancies increase very fast on approaching the separation point (for circular cylinder).*

- *The circumstance that real flows can support considerable rates of pressure increase (adverse pressure gradients) in a large number of cases without separation is due to the fact that the flow is mostly turbulent. The best known examples include cases of flow past circular cylinders and spheres, when* *separation occurs much further upstream in laminar than in turbulent flow. It is nevertheless useful to consider laminar flow because it is much more amenable to mathematical treatment than is the case of turbulent flow....At the present time these very complicated phenomena (separation in turbulent flow) are far from being understood completely...*

- *The form drag which does not exist in frictionless subsonic flow, is due to the fact that the presence of the boundary layer modifies the pressure distribution on the body as compared with ideal flow, but its computation is very difficult.*

- *The origin of pressure drag lies in the fact that the boundary layer exerts a displacement action on the external stream. This modifies somewhat the pressure distribution on the body surface. In contrast with potential flow (d'Alembert's paradox), the resultant of this pressure distribution modified by friction no longer vanishes but produces a preessure drag which must be added to skin friction. The two together give form drag.*

- *In the case of the most important fluids, namely water and air, the viscosity is very small and, consequently, the forces due to viscous friction are, generally speaking, very small compared with the remaining forces (gravity and pressure forces). For this resaon it was very difficult to comprehend that very small frictional forces omitted in classical (inviscid) theory influenced the motion of a fluid to so large extent.*

Prandtl described the difficulties himself in *Applied Hydro- and Aeromechanics* from 1934:

- *Only in the case where the "boundary layer" formed under the influence of the viscosity remains in contact with the body, can an approximation of the actual fluid motion by means of a theory in terms of the ideal frictionsless fluid be attempted, whereas in all cases where the boundary leaves the body, a theoretical treatment leads to results which do not coincide at all with experiment. And it had to be confessed that the latter case occurs most frequently.*

In a nutshell, these quotes present much of the essence of modern fluid mechanics propagated in standard books and courses in fluid mechanics: Drag and lift in slightly viscous flow are claimed to arise from separation in a thin viscous laminar boundary layer brought to stagnation with reversed flow due to an adverse pressure gradient. On the other hand, both Prandtl and Schlichting admit that this standard scenario does not describe turbulent flow, always arising in slightly viscous flow, but persists that *"it is nevertheless useful to consider laminar flow because it is much more amenable to mathematical treatment"*. However, turbulent and laminar flow have different properties, and drawing conclusions about turbulent flow from studies of laminar flow can be grossly misleading.

38.3 Prandtl's Resolution of d'Alembert's Paradox

The commonly accepted resolution of d'Alembert's Paradox propagated in the fluid dynamics literature is attributed to Prandtl, who in his 1904 article suggested that drag/lift possibly could result from *transversal vorticity* caused by tripping of the flow by a no-slip boundary condition and thereby changing the global flow. Prandtl was inspired by Saint-Venant stating in 1846 [85]:

- *But one finds another result (non-zero drag) if, instead of an inviscid fluid – object of the calculations of the geometers Euler of the last century – one uses a real fluid, composed of a finite number of molecules and exerting in its state of motion unequal pressure forces having components tangential to the surface elements through which they act; components to which we refer as the friction of the fluid, a name which has been given to them since Descartes and Newton until Venturi.*

Saint-Venant and Prandtl thus suggested that drag in a real fluid possibly could result from tangential frictional forces in a thin viscous boundary layer creating transversal vorticity, and accordingly inviscid potential flow could be discarded because it has no boundary layer. These suggestions have over time been transformed to become an accepted fact of modern fluid dynamics, questioned by few. The mathematician *Garret Birkhoff* (1911-1996) conjectured in [92] that drag instead could be the result of an instability of potential flow, but after a devastating review [86], Birkhoff did not pursue this line of thought, and even partly changed position in a second edition of the book.

But we shall see that Birkhoff was correct: Potential flow with zero drag is unstable, and this is the reason it cannot be observed. What can be observed is turbulent flow with substantial drag, and there are aspects or outputs of turbulent flow which are *wellposed* in the sense that they do not change under small perturbations, while potential flow is *illposed* with respect to all outputs of any significance and thus unphysical.

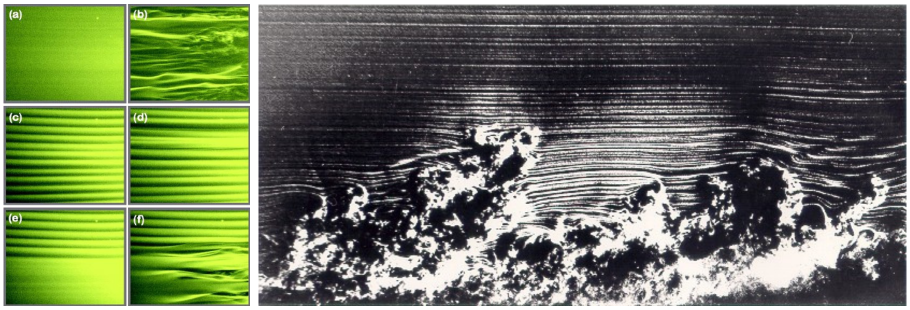

Figure 38.2: Horisontal boundary layers with streaks of streamwise vorticity (top view above), and turbulent boundary layer (side view below).

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